

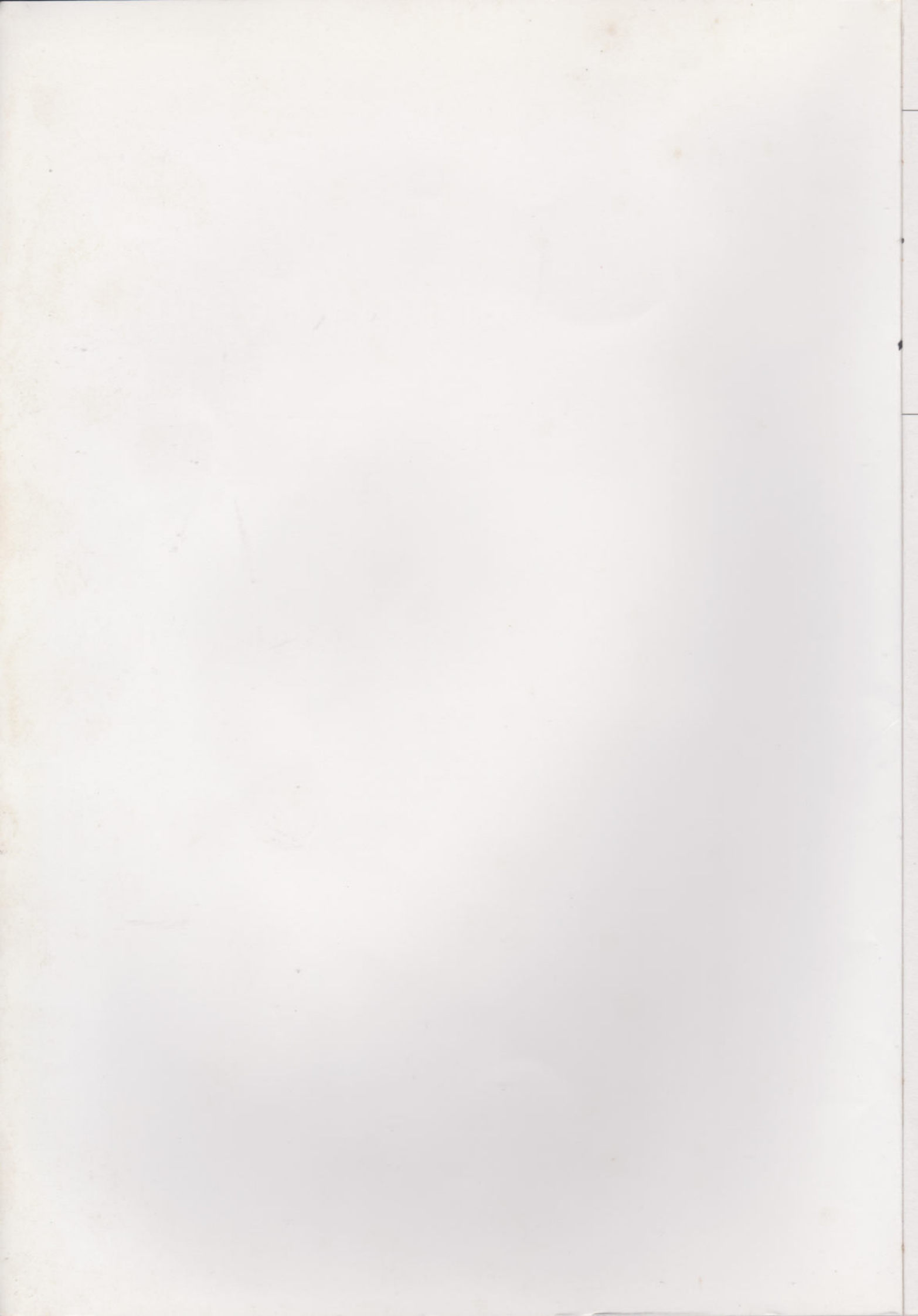


The Open University

MU120
Open Mathematics

Preparatory Resource Book A

Modules 1–4





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How to use this book

Read this first

This is a resource book. It has been written for you to use as and when you need to revise topics. Use the material in conjunction with the unit *Preparing for Open Mathematics*, which gives advice on which modules you may find it helpful to study.

This book has four modules (or chapters), on the topics listed in the Contents List, p.3. Each module has three sections. All sections have the same structure:

First some diagnostic questions headed ‘Try these first’. Tackle all of these so that you know where to focus your efforts. The answers follow the questions and tell you which subsection to study in more detail if you found the question on that topic difficult.

Then the subsections follow, each focusing on a topic identified in the diagnostic questions. You need not study a subsection in detail if you feel reasonably confident about the topic. Each subsection explains the topic and has a number of worked examples.

At the end of each subsection are exercises headed ‘Try some yourself’. These give you practice in applying what you have just learned or revised, and include exercises for reflecting on your learning and putting ideas into words. Aim to do these exercises as soon as you have studied that subsection. They are part of your learning process. Solutions to these exercises are at the end of the book.



Some of the examples and exercises in this book require the use of a calculator. They are marked with a ‘calculator icon’ in the margin. Other examples and exercises may be tackled whether or not you have access to a calculator yet. Instructions covering all the calculator techniques you need here are in Chapter 1 of the *Calculator Book*. You may wish to defer calculator exercises in this resource book until you have looked at it and indeed until you have acquired a calculator. But do begin work on the rest of this book! Once the course itself starts, you should have your own course calculator so the ‘calculator icon’ will not appear.

This is the first of two preparatory resource books. The second, *Preparatory Resource Book B*, has a further three modules. Gradually working through the material in these books will build up your confidence, we hope, in your basic skills ready for the start of Open Mathematics.

You can come back to this *resource* book at any time during your study of MU120 if you find you need to revise something.

Module 1 Numbers, units and arithmetic

MU120 assumes that you are familiar with whole numbers, decimals and fractions, both positive and negative. You should know when and how to add, subtract, multiply and divide. Although you will have a course calculator, you will still need to do simple calculations in your head or on paper. Such calculations could be to estimate the size of an answer, as a check for your calculator work.

1.1 The size of numbers

Numbers are used to specify a quantity or amount. For example, people give their ages as a number of years: 'I am 51 years old' or 'I am five and a half'. Votes in elections may be described in thousands. Temperatures are measured in positive and negative degrees Celsius. Numbers smaller than one may be expressed as fractions or decimals. This section considers whole numbers, decimals, fractions and negative numbers and reminds you of the essence of a number—how 'big' it is.

Try these first

These are diagnostic questions. Try them to see which topics you need to revise.

- 1 (a) Write 'twenty thousand one hundred and forty-four' as a number.
(b) Say (or write) the number 31 002 103 in words.
(c) Place the following numbers of votes in an election in order, with the smallest first:

9999 5001 49 020

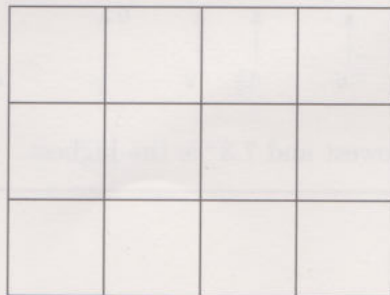
- (d) Multiply 9494 by 100.

- 2 (a) Place the following numbers of kilometres in order, with the smallest distance first:

1.013 1.103 1.0103 1.0129

- (b) Divide 202.15 by 1000. What is 202.15 metres in kilometres?

- 3 (a) The diagram below shows an oatmeal cake marked into 12 equal portions. I want to give my sister a third of the cake. Show on the diagram where I could cut the cake, and shade in what is left over.

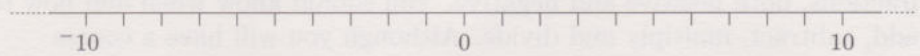


- (b) Which two of the following fractions are equivalent to each other?

$\frac{7}{8}$ $\frac{49}{64}$ $\frac{35}{40}$ $\frac{8}{9}$

- 4 Place the following temperatures (in degrees Celsius) on the number line and hence say which is the lowest and which the highest.

$$-2^{\circ} \quad 7^{\circ} \quad -6^{\circ} \quad 7.3^{\circ} \quad \frac{1}{2}^{\circ} \quad -3\frac{1}{2}^{\circ} \quad 3\frac{1}{2}^{\circ}$$

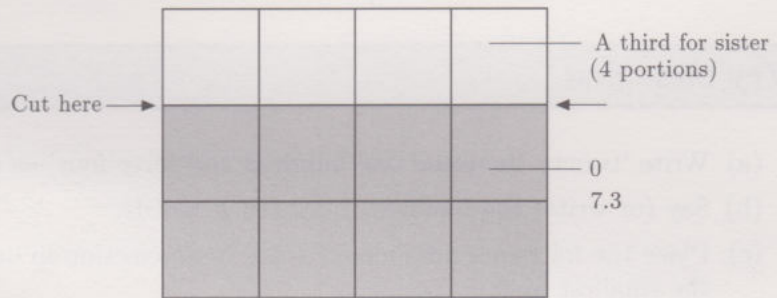


Check your answers

- Follow-up in Section 1.1.1 1 (a) The number is 20 144 (or it could be written 20,144).
 (b) Thirty-one million two thousand one hundred and three.
 (c) 5001, 9999, 49 020.
 (d) 949 400.

- Follow-up in Section 1.1.2 2 (a) 1.0103, 1.0129, 1.013, 1.103.
 (b) 0.202 15. So 202.15 metres are 0.202 15 kilometres.

- Follow-up in Section 1.1.3 3 (a)

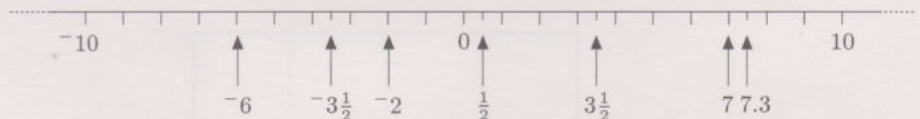


(There are several other ways of cutting the cake into one-third (4 portions) and two-thirds (8 portions). Whichever way you choose, you should have shaded 8 portions.)

- (b) Only $\frac{7}{8}$ and $\frac{35}{40}$ are equivalent fractions, since $\frac{7 \times 5}{8 \times 5} = \frac{35}{40}$.

(Notice that neither $\frac{49}{64}$, which is $\frac{7 \times 7}{8 \times 8}$, nor $\frac{8}{9}$, which is $\frac{7+1}{8+1}$, are equivalent to $\frac{7}{8}$.)

- Follow-up in Section 1.1.4 4 In order they are: -6° , $-3\frac{1}{2}^{\circ}$, -2° , $\frac{1}{2}^{\circ}$, $3\frac{1}{2}^{\circ}$, 7° , 7.3° .



So -6° is the lowest and 7.3° is the highest.

1.1.1 Whole numbers

Whole numbers arise from counting: for example the number of sheep in a field or the number of votes in an election.

Our everyday number system is the *decimal system*, where the position of a digit within the number determines whether it represents units, tens, hundreds, thousands etc. For example, the number 1375 means one thousand three hundred and seventy-five. The position of the 3, third from the right, means that it represents 3 hundreds.

A thousand thousand, 1000 000, is called a million and a thousand million, 1000 000 000, is called a billion. To compare two numbers, it sometimes helps to write them, or think of them, in columns:

Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units
----------	----------------------	------------------	-----------	----------	------	-------

The position of a digit in the columns is called its **place value**.

A British billion used to be a million million, but now the US convention of a thousand million is normally used.

Example 1

A lottery organiser announces that this week's winnings will be over two million pounds. After the draw, the organisers announce that the winnings were £2 201 995. Was the announcement correct?

Solution

The issue is whether the figures represent a number greater than two million, or not. To test this, write the two numbers in columns, one below the other.

Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units
2	0	0	0	0	0	0
2	2	0	1	9	9	5

Now look at the numbers *from the left*. Both numbers have a 2 in the millions column, so move to the next place. The first number has a 0 in the next column (i.e. hundred thousands) whereas the second number has a 2. So the second number is larger, the announcement was correct and more than two million pounds was paid out.

In practice you probably won't want to write the headings in the columns each time. But do keep their meanings in mind.

Writing numbers in place-value columns gives an easy way of multiplying by 10, 100, 1000 and so on. Have a look at the place value table again.

Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units
----------	----------------------	------------------	-----------	----------	------	-------

The headings of the columns are 'powers of ten'—this is explained in Module 4.

Notice that moving all the digits one column to the left in the table (and adding a zero to the units column) is the same as multiplying by ten; for instance, ten tens are a hundred, ten hundreds are a thousand. If you move *two* columns to the left, you multiply by a hundred; for instance, a hundred tens are a thousand. Moving *three* columns to the left is equivalent to multiplying by a thousand, and so on.

Example 2

After local government re-organization a new council expects to receive cash payments in ten instalments from about 26 000 households. In order to negotiate the cash collection at post offices it needs to know how many payments are made altogether in a year. How many are there likely to be?

Solution

The total number of payments is the number of households times the number of payments per household, i.e. $26\,000 \times 10$. This can be worked out by moving all the digits of 26 000 one place to the left, and adding a zero at the end:

M	H Th	T Th	Th	H	T	U	$\times 10$
=	2	6	0	0	0	0	

Digits moved one place to the left

Extra zero added

So $26\,000 \times 10 = 260\,000$.

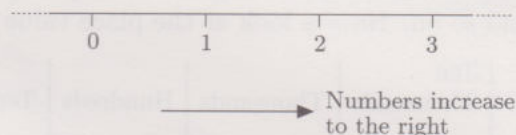
A total of 260 000 cash payments should be expected.

Solutions on page 96.

Try some yourself (1.1.1)

- 1 Write each of the following three numbers in numerals and then place them in ascending order:
 - eight hundred and eight thousand
 - two million and twenty-four
 - nine thousand nine hundred and ninety-eight
- 2 Multiply 2490 by 100.

It is often useful in mathematics to think of numbers stretched out along the imaginary **number line**. The diagram below shows part of the number line and the numbers 0, 1, 2 and 3.



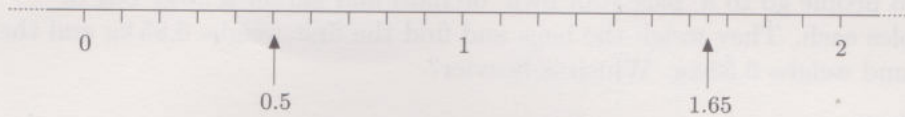
As you move to the right along the number line, the numbers increase.

This concept will be useful for picturing other numbers, as well as whole numbers.

1.1.2 Decimals

Quantities can be smaller than one (such as 0.5 kg) or take values between whole numbers (such as a height of 1.65 metres). Numbers smaller than one are expressed as decimals or as fractions. Decimals are often easier to work with (especially when using a calculator). Decimals are in this subsection, and fractions in the next (1.1.3).

Decimals can be indicated on the number line in between whole numbers. 0.5 and 1.65 are indicated on the figure below.



Decimals arise when whole numbers are divided by ten, a hundred etc.

Example 3

What is 3579 cents in dollars? (There are 100 cents in a dollar (\$).)

Solution

You need to divide 3579 by 100. To multiply a number by 100, you shift the digits two places to the left (and add zeros to fill the spaces). To reverse the process, i.e. to divide by 100, move the digits two places to the right. To show this, you need to mark the end of the units column, and this is done by means of a **decimal point**.

$$\begin{array}{c|c|c|c|c}
 \text{Th} & \text{H} & \text{T} & \text{U} & \\
 \hline
 3 & 5 & 7 & 9 & \cdot \\
 \hline
 = & & 3 & 5 & \cdot 79
 \end{array} \div 100$$

$\xrightarrow{\hspace{2cm}}$
 Digits moved two places to right

So the answer is 35.79, or in words, thirty five **point** seven nine (*not* seventy nine).

So 3579 cents = \$35.79.

What do the digits to the right of the decimal point mean? The first column to the right of the decimal point is obtained by dividing the units by 10; this is the 'tenths' column. Similarly the next column contains the hundredths, and so on. So here is the extended place value table:

...	Thousands	Hundreds	Tens	Units	Tenths	Hundredths	Thousandths	...
-----	-----------	----------	------	-------	--------	------------	-------------	-----

A tenth is 0.1 or 1 divided by 10, i.e. $\frac{1}{10}$.

A hundredth is 0.01 or 1 divided by 100, i.e. $\frac{1}{100}$.

A thousandth is 0.001 or 1 divided by 1000, i.e. $\frac{1}{1000}$.

So 0.101 is one tenth plus one thousandth or $\frac{1}{10} + \frac{1}{1000}$.

Notice that you need to include the zeros after the decimal point in numbers like 0.05 or 0.005, just as you do before the decimal point in

numbers like 50 or 500, in order to indicate the place value of the 5. Also, note that often there is a 0 before the decimal point for numbers smaller than one. Whether it is written as 0.1 or .1 the value is still the same (one tenth). 0.1 is more usual when writing, .1 is more usual on calculators. In general, adding the zero before the decimal point is a clearer way of writing the number – the decimal point will then not be missed.

Example 4

Two people go to a 'pick your own' orchard and gather a large bag of apples each. They weigh the bags and find the first weighs 6.85 kg and the second weighs 6.58 kg. Which is heavier?

Solution

The point at issue is which number is larger, 6.85 or 6.58. To determine this, write the digits in a place value table and then compare the values in each column:

Units	Tenths	Hundredths
6	8	5
6	5	8

Again, work from the left. The digits in the units column are the same, so look at the next column. The first number has 8 tenths and the second number has 5 in this column (5 tenths), so the first number is bigger which means that the first bag is heavier.

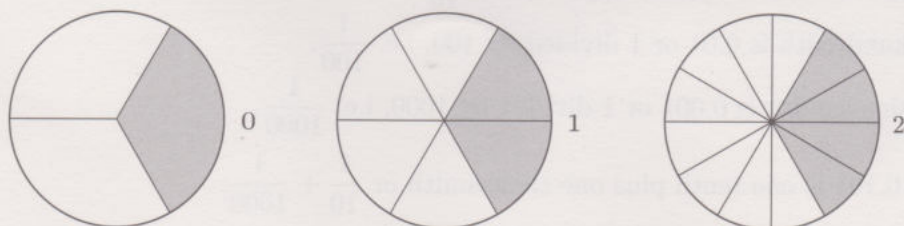
Solutions on page 96.

Try some yourself (1.1.2)

- 1 What is 370.76 grams in kilograms? There are 1000 grams in a kilogram.
- 2 Place the following measurements in ascending order:
0.1704 metres 0.1074 metres 0.0714 metres 0.0741 metres

1.1.3 Fractions

A fraction is written as one number over another (such as $\frac{3}{10}$) and means the top number divided by the bottom number. The top number, 3, is called the **numerator** and the bottom number, 10, is called the **denominator**. Whole numbers can be written as fractions with the denominator 1. Thus, 2 can be written as $\frac{2}{1}$. The same fraction may be written in different ways, depending upon the context.



In these circles, all of the shaded areas represent the same fraction of the circle: one third of it. So

$$\frac{1}{3} = \frac{2}{6} = \frac{4}{12}.$$

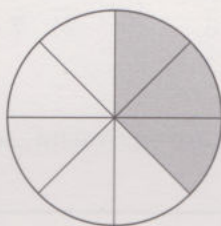
Fractions which are equal in value but are written with different numerators and denominators are called **equivalent fractions**. You can find equivalent fractions for any given fraction by multiplying top and bottom by the same whole number. The fraction $\frac{1}{3}$, is the *simplest form* of all its equivalent fractions, because it cannot be 'simplified' further (by dividing top and bottom by the same whole number); $\frac{2}{6}$ or $\frac{4}{12}$, on the other hand, can be simplified to $\frac{1}{3}$ by dividing top and bottom by 2 and 4 respectively.

Decimals are themselves a special form of fractions (0.3 is $\frac{3}{10}$). There are situations where it is simpler to use fractions. For example, if you wanted to divide a circle into six equal pieces, you think of the fraction $\frac{1}{6}$. There is no exact decimal equivalent of $\frac{1}{6}$. Recipes often ask for fractions, such as 'half a pint of milk' or a cooking time of 'three-quarters of an hour'.

$\frac{1}{6}$ has no exact decimal equivalent because dividing 6 into 1.0000... produces a never-ending string of 6s: 0.1666....

Example 5

What fraction of the circle is shaded in the diagram below?



Solution

The fraction is $\frac{\text{number of shaded parts}}{\text{total number of parts}}$. The circle is divided into 8 equal sections, so each section is $\frac{1}{8}$. Three of them are shaded so the shaded part is $\frac{3}{8}$ of the circle.

Example 6

- Find some equivalent fractions for $\frac{1}{4}$.
- What is the simplest form of $\frac{15}{20}$?

Solution

(a) Some examples are $\frac{2 \times 1}{2 \times 4} = \frac{2}{8}$ $\frac{3 \times 1}{3 \times 4} = \frac{3}{12}$ $\frac{25 \times 1}{25 \times 4} = \frac{25}{100}$

- (b) Top and bottom are divisible by 5. So

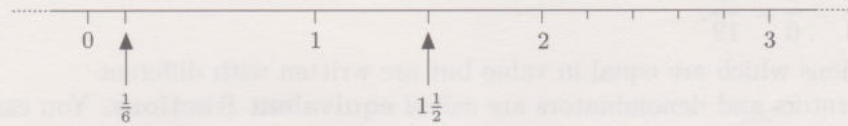
$$\frac{15}{20} = \frac{15 \div 5}{20 \div 5} = \frac{3}{4}$$

3 and 4 have no common factors so $\frac{3}{4}$ is the simplest form.

Multiply or divide top and bottom by the *same* whole number to give an equivalent fraction.

Note $1\frac{1}{2}$ may also be written as 1.5 or as $\frac{3}{2}$.

You can indicate fractions along the number line by marking points between the whole numbers. The numbers $\frac{1}{6}$ and $1\frac{1}{2}$ are shown below.



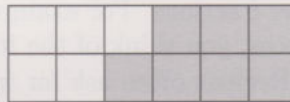
Solutions on page 96.

Try some yourself (1.1.3)

- 1 For each of the following shapes, indicate the fraction of the whole shape represented by the shaded parts:



(b)



(c)



- 2 (a) Which of the following are equivalent fractions for $\frac{2}{3}$?

(i) $\frac{4}{6}$ (ii) $\frac{3}{4}$ (iii) $\frac{12}{18}$ (iv) $\frac{7}{10}$

- (b) Which of the following are pairs of equivalent fractions?

(i) $\frac{5}{8}, \frac{10}{16}$ (ii) $\frac{3}{4}, \frac{5}{6}$ (iii) $\frac{12}{21}, \frac{4}{7}$ (iv) $\frac{5}{100}, \frac{1}{20}$

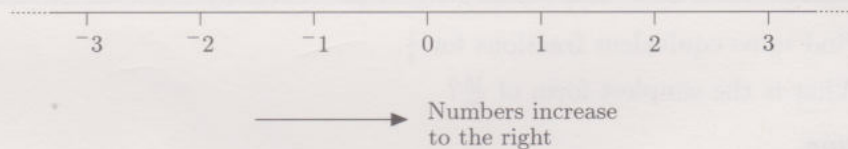
(v) $\frac{1}{3}, \frac{3}{5}$

For those pairs which are equivalent, say which is in its simplest form.

1.1.4 Negative numbers

Numbers can be positive or negative, i.e. greater than or less than zero. Negative numbers have several uses; for example, to measure temperatures below zero, such as -3°C ('minus 3 degrees Celsius'). They are also used to represent debts and overdrawn accounts: a bank balance of $-\text{£}84.33$ means 'overdrawn by $\text{£}84.33$ '.

Negative numbers are shown on the number line to the left of 0. The diagram below shows -1 , -2 and -3 .



The numbers always increase as you move to the right along the number line, *wherever you start from*. 1 is to the right of -2 , so 1 is greater than -2 . 1° is warmer than -2° and a bank balance of $\text{£}1$ is more than one of $-\text{£}2$ (two pounds overdrawn).

It is worth mentioning notation at this point. You may have noticed that the minus sign used to denote a negative number is shorter, closer to the number and raised, compared with the minus sign used to denote subtraction. It is important to distinguish between the two, and it can help to think of -3 , say, as 'negative 3' rather than 'minus 3'. In the

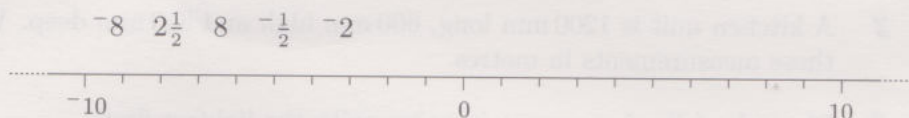
calculation $8 - 3$ (8 subtract 3), the minus sign is an *operator*, an instruction to subtract; in -3 , the sign is part of the number. The two types of minus sign are distinguished on many calculators, including the MU120 course calculator. If you have a calculator handy, look at the key pad to see if you have two keys. They are often labelled $\boxed{-}$ and $\boxed{(-)}$.



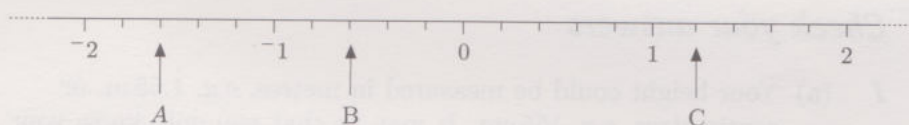
Try some yourself (1.1.4)

Solutions on page 96.

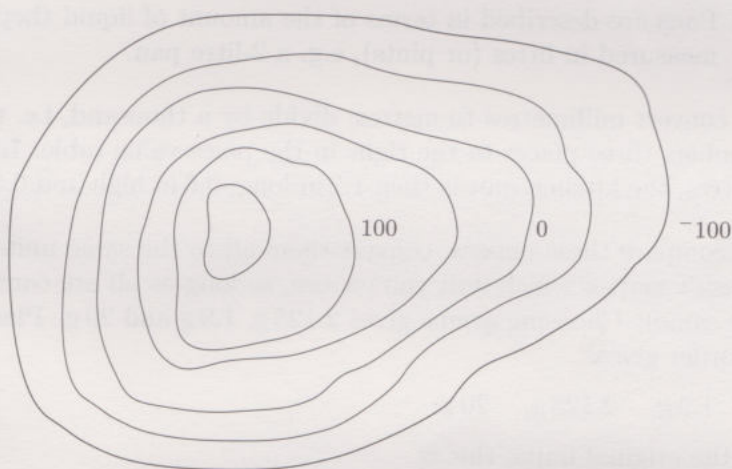
- 1 (a) Mark the following points on the number line below:



- (b) What are the numbers marked A , B and C on the number line below?



- 2 Contour lines on a map show all the points at a given height above sea level. The lines are drawn for each height at 50-metre intervals, and points below sea level are shown by negative heights. The diagram below shows a peak with six such contour lines. The lines for -100 , 0 and 100 have been labelled. Label the other three contour lines.



1.2 Units of measurement

Numbers are used to count things (such as 54 sheep in a field) or, together with *units of measurement*, to quantify something (such as a 2-litre water bottle or a $2\frac{1}{2}$ -year waiting list for an operation). This section reminds you of some units of measurement and considers appropriate units to use. We also need to make comparisons between quantities given in different units.

These are diagnostic questions. Try them to see which topics you need to revise.

There are 1000 mm (millimetres) in a metre.

mg means milligram:
 $1000 \text{ mg} = 1 \text{ g}$.
 kg means kilogram:
 $1000 \text{ g} = 1 \text{ kg}$.

Try these first

- 1 Choose appropriate units for each of the following, using metric units where possible:
 - (a) your height;
 - (b) your weight;
 - (c) the length of time to boil an egg;
 - (d) the capacity of a large pan.
- 2 A kitchen unit is 1200 mm long, 600 mm high and 350 mm deep. Write these measurements in metres.
- 3 Place the following masses in order, with the lightest first:
 2125 mg 1.9 g 0.02 kg

Check your answers

- Follow-up in Section 1.2.1 1 (a) Your height could be measured in metres, e.g. 1.55 m, or centimetres, e.g. 155 cm. It may be that you only know your height in feet and inches, e.g. 5 ft 1 in.
- (b) Your weight might be given in kilograms, e.g. 58.06 kg, or perhaps in stones and pounds, e.g. 9 st 2 lb.
- (c) The length of time to boil an egg is usually given in minutes, e.g. 3 min.
- (d) Pans are described in terms of the amount of liquid they hold, measured in litres (or pints), e.g. a 2-litre pan.
- Follow-up in Section 1.2.2 2 To convert millimetres to metres, divide by a thousand, i.e. move the numbers three places to the right in the place value table. In terms of metres, the kitchen unit is then 1.2 m long, 0.6 m high and 0.35 m deep.
- Follow-up in Section 1.2.3 3 To compare these masses, convert them all to the same units (it doesn't matter which unit you choose, so long as all are converted to the same). Choosing grams gives 2.125 g, 1.9 g and 20 g. Placing these in order gives:
- 1.9 g, 2.125 g, 20 g.
- In the original units, this is
- 1.9 g, 2125 mg, 0.02 kg.

1.2.1 Which units to use

It is important to choose appropriate units, both to have a sense of the size you are talking about and also to avoid the awkwardness of very large or very small numbers.

To measure time, you use different units according to the context: minutes and seconds for telephone calls; hours for a train journey; days or weeks for a holiday; months or years for the age of a child.

The money system used in this course is the UK decimal system:

£1 = 100 p (one pound is equivalent to 100 pence)

Generally you use pence for prices less than a pound: 50p rather than £0.50. Other currencies may also be used e.g. when considering conversion between currencies.

For distance, mass and liquid measures, MU120 usually uses the *metric system* of units. For example, the distance a cyclist cycles in a day is likely to be given in kilometres (abbreviated to km), a person's height in metres (m) and a waist measurement in centimetres (cm). The prefixes 'kilo' and 'centi', together with 'deci' and 'milli', are used throughout the metric system:

kilo is 1000, so a kilometre (km) is 1000 metres;

deci is $\frac{1}{10}$, so a decimetre (dm) is $\frac{1}{10}$ metre, or 0.1 m (there are 10 decimetres in a metre);

centi is $\frac{1}{100}$, so a centimetre (cm) is $\frac{1}{100}$ metre, or 0.01 m (there are 100 centimetres in a metre);

milli is $\frac{1}{1000}$, so a millimetre (mm) is $\frac{1}{1000}$ metre, or 0.001 m (there are 1000 millimetres in a metre).

For mass, kilograms (kg), grams (g) or milligrams (mg) are usually used and a metric tonne for very large masses (1000 kg). For liquid capacity, litres (l), decilitres (dl), centilitres (cl) or millilitres (ml) are used.

Full details of the relationships linking these units are given on a separate blue handbook sheet for you to keep in your learning file.

Occasionally common imperial units are used: miles, pints etc.

The word 'mass' is scientifically more precise than 'weight', although people usually talk about weight rather than mass.

Example 7

Imagine a friend is planning a new kitchen in her house. In the kitchen showroom she noticed that the measurements of most of the kitchen units were given in millimetres. One worktop, for instance, is 575 mm deep. What units should she use to measure the large room in the house where the new kitchen will be located?

Solution

She could measure in millimetres, but that would give large numbers. It is difficult to visualize the room in terms of millimetres—1 mm is approximately the thickness of a piece of wire. Can you imagine how many of those would fit along one wall! But thinking of it in terms of metres may be easier. A single bed is about one metre wide so you could visualise how many would fit side by side along each wall. This will help check measurements. So she might measure each side of the room in metres (e.g. 2.82 m by 4.25 m). To avoid using decimals, she could have used centimetres instead (giving 282 cm by 425 cm). Either of these would be sensible. It would be unhelpful (to say the least!) to quote the dimensions in kilometres (e.g. 0.002 82 km by 0.004 25 km).

Solutions on page 96.

Try some yourself (1.2.1)

- 1 Suggest appropriate units for each of the following:
 - (a) the age of the kitten when it is weaned;
 - (b) the distance between one train station and the next;
 - (c) the amount of one 'active' ingredient of a prescribed pill;
 - (d) the cost of leaving the light on continuously for 24 hours.

1.2.2 Converting units

A great advantage of the metric system of units is that conversion between units within the system is particularly easy. For example, '£1 is worth 100 p' is converting one pound into pence. To convert pounds to pence, you multiply by 100. So £2 is 200 p, and £2.63 is 263 p.

To convert from pence to pounds, you need to reverse this process, i.e. to *divide* by 100 (moving the digits two places to the right) so that 845 p becomes £8.45. In the same way, any of the metric units can be converted simply by multiplying or dividing by ten, a hundred or a thousand. For example, the statement '1 m = 1000 mm' tells you that to convert from metres to millimetres you multiply by 1000.

The general rule when converting units is: to convert from a larger unit (e.g. metres) to a smaller one (e.g. millimetres), *multiply*; to convert from a smaller unit to a larger one, *divide*.

Example 8

The measurements of the kitchen units in the example above were given in millimetres—a unit was 575 mm deep. What is this in metres?

Solution

To convert millimetres to metres, divide by 1000. So to convert 575 mm to metres, move the digits three places to the right:

$$575.0 \div 1000 = 0.575.$$

So the answer is 0.575 m.

There is no handy metric system for time. There are 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day and 7 days in a week; a month may be 28, 29, 30 or 31 days long and a year may contain 365 or 366 days.

Try some yourself (1.2.2)

Solutions on page 96.

- 1 A person's height is given as 1.65 m. What is this in centimetres?
- 2 (a) Express 675 mm in centimetres.
 (b) Express 45.2 km in metres.
 (c) Express 3.5 kg in grams.
 (d) Express 167.2 mm in metres.

1.2.3 Comparing measurements

In order to compare quantities, it is best to express them in the same units.

Example 9

Three children have just measured their own heights in metric units. Isaac says 'My height is 1098', Jasmine says 'My height is 112' and Kim says 'Mine is 1.1'. What units were they using? Who is the tallest?

Solution

First decide what units the children were using and then convert all the measurements to the same units. It is reasonable to guess that the heights are between 1 and 2 metres. (A height between 10 cm and 20 cm would mean they were not yet born, whereas a height between 10 m and 20 m would make them as tall as a house!) So Isaac has given his height in millimetres and his height in metres is $1098 \div 1000 = 1.098$ m. Jasmine has given her height in centimetres, so her height in metres is $112 \div 100 = 1.12$ m. Kim's height is in metres already, so Kim is 1.1 m tall. So, the heights in metres are:

Isaac 1.098;
 Jasmine 1.120;
 Kim 1.100.

Comparing from the left, all have 1s in the units column. Isaac has a 0 in the tenths column, but Jasmine and Kim have 1s in the tenths column. So Jasmine and Kim are both taller than Isaac. In the hundredths column, Jasmine has 2 and Kim has 0. So Jasmine is the tallest.

Inserting a 0 to the right of Jasmine's 1.12, and two 0s to the right of Kim's 1.1, has no effect on the value but is useful for comparisons as here.

Try some yourself (1.2.3)

Solutions on page 97.

- 1 Place each of the following sets of quantities in order, smallest first.
 (a) X = 0.5 l Y = 490 cl Z = 510 ml
 (b) X = 20 p Y = £0.02
 (c) X = 0.039 kg Y = 4 g Z = 30 000 mg

1.3 Arithmetic

Although you have a calculator at your disposal, you need to be able to carry out some calculations (usually with simpler numbers) in order to check your calculator work. Actually handling numbers, on paper or in your head, helps you to understand the nature of the calculations you perform on your calculator and gives you a deeper understanding of the underlying mathematical processes. In some cases, it really is quicker and easier to carry out a calculation by hand rather than key it into your calculator. However, do use your calculator for more cumbersome arithmetic.

The arithmetic system works not only for positive whole numbers but also for fractions, decimals and negative numbers. It can also be used when letters represent unknown numbers (algebra). It is therefore important to understand the principles for whole numbers, so you can extend these ideas to include different types of numbers.

These are diagnostic questions. Try them to see which topics you need to revise.

Try these first

- 1 Try these in your head, or on paper, rather than using a calculator:
 - (a) A doctor prescribes 3 pills a day and the prescription is to last for 4 weeks. How many pills should the pharmacist supply?
 - (b) The target score in a game of darts is obtained by subtracting the score of the three darts thrown in a player's turn from the current target score on the board. In the game of '301', the target score starts at 301. If a player's first three darts are a double 19, a single 20 and a treble 17, what is the new target score?
 - (c) Apples are sold in a pack of 7 for 95 p or singly 14 p each. Which is the better buy?
- 2
 - (a) Suppose you went shopping with a friend and your combined bill came to £85. You knew that you had spent £47.65. What was your friend's share of the bill?
 - (b) Evaluate each of the following.
 - (i) 7.9×0.8 (ii) 82.3×40
 - (iii) $7.20 \div 0.8$ (iv) $62.30 \div 40$
 - (c) Use your answer to (b) to evaluate the following.
 - (i) Special cheese is £7.90 per kilo. How much does 0.8 kilos cost?
 - (ii) Petrol is 82.3 p per litre. How much does 40 litres cost?
 - (iii) The till receipt showed the cheese cost £7.20, and you bought 0.8 kilos. How much was it per kilo?
 - (iv) The bill was £62.30 for 40 small children's party hats. How much were they each? Was there an error in the bill or a discount?

- 3 For each of the following statements, say whether it is true or false.
- (a) Multiplying one number by a second number always gives an answer greater than the first number.
 - (b) When adding three numbers together, it doesn't matter in which order you add them.
 - (c) When carrying out an addition and a multiplication, such as $2 + 3 \times 4$, it doesn't matter whether you do the addition or the multiplication first.
 - (d) Putting brackets into an expression can change the order of the calculations.
 - (e) Adding the same number to the top and bottom of a fraction gives you an equivalent fraction.
- 4
- (a) If you divide a cake into 12 equal pieces and then eat 3 pieces, what fraction of the cake is left?
 - (b) A recipe specifies a cooking time of $\frac{3}{4}$ hour and suggests checking and basting $\frac{2}{3}$ of the way through the cooking time. After how long should you check it?
- 5
- (a) If you had £3 in your bank account and drew out £10, how much would you have left?
 - (b) The temperature was -5°C on Monday and dropped overnight by 6°C . What was the temperature on Tuesday morning?
 - (c) Evaluate each of the following.
 - (i) $-3 + -12$ (ii) $-4 - -11$ (iii) 7×-6 (iv) $-56 \div -7$
 Think of a financial context where each might be an appropriate calculation (bear in mind that negative numbers can represent debts).

Check your answers

- 1
- (a) There are 7 days in a week, so 4 weeks is $7 \times 4 = 28$ days. The number of pills required is $28 \times 3 = 84$. Section 1.3.1
 - (b) The player scores 2×19 , 20 and 3×17 , which gives $38 + 20 + 51 = 109$. $301 - 109$ gives a new target score of 192.
 - (c) The apples in packs cost $\frac{95}{7}$ p each. Dividing 95 by 7 gives $13 \frac{4}{7}$, which is less than the 14 p per apple sold singly. Alternatively, you could compare the cost of 7 apples: 7 apples at 14 p each would cost 98 p, which is more than 95 p. So the apples in the pack would appear to be the better buy. (However, if you do not want 7 apples, and are likely to leave some of them to rot, you would be better off just buying the number of apples you want!)

- Section 1.3.3 **2** (a) Your friend's share of the bill is $\text{£}85 - \text{£}47.65 = \text{£}37.35$. There are many ways to do this subtraction. Setting out as a formal sum is one approach, but counting on (as an assistant may do to give change) is another (from $\text{£}47.65$ a further 35 p is needed to make $\text{£}48$, then $\text{£}2$ to make $\text{£}50$ and $\text{£}35$ to make $\text{£}85$ gives $\text{£}37.35$).
- (b) (i) $7.9 \times 0.8 = 6.32$ (multiply by 8 then shift the digits one place to the right).
- (ii) $82.3 \times 40 = 3292$ (multiply by 4 then shift the digits one place to the left).
- (iii) $7.2 \div 0.8 = \frac{7.2}{0.8} = \frac{7.2 \times 10}{0.8 \times 10} = \frac{72}{8} = 9$.
- (iv) $62.3 \div 40 = \frac{62.3}{40} = \frac{6.23}{4} = 1.5575$.
- (c) (i) 0.8 kilos of cheese costs $\text{£}6.32$.
- (ii) 40 litres of petrol costs $3292 \text{ p} = \text{£}32.92$.
- (iii) The cheese costs $\text{£}9$ per kilo.
- (iv) The division sum makes them at $\text{£}1.5575$ each, but they must be a whole number of pence each so only two decimal places are meaningful. Hence there was either an error or some discount.

- Section 1.3.3 **3** (a) False. For example, multiplying 7.9×0.8 in Question 2(b)(i) gave 6.32, which is less than 7.9. In general multiplying a positive number by a number less than one gives a smaller number.

- Section 1.3.2 (b) True. For example $2 + 3 + 4$ is the same as $4 + 2 + 3$.

- Section 1.3.2 (c) False. Carrying out the addition first gives $2 + 3 = 5$, then multiplying by 4 gives 20. Carrying out the multiplication first gives $3 \times 4 = 12$, then adding to 2 gives 14 (which is the correct procedure here).

- Section 1.3.2 (d) True. For example writing brackets around $2 + 3$ in the expression $2 + 3 \times 4$ gives $(2 + 3) \times 4 = 20$, whereas without the brackets you should carry out the multiplication first to give 14.

- Sections 1.3.2 and 1.3.4 (e) False. For example adding 2 to the top and bottom of $\frac{1}{2}$ gives $\frac{3}{4}$. But $\frac{3}{4}$ is not equivalent to $\frac{1}{2}$.

- Section 1.3.4 **4** (a) You have eaten $3 \times \frac{1}{12} = \frac{3}{12}$ (or $\frac{1}{4}$) of the cake. So there is $\frac{9}{12}$ (or $\frac{3}{4}$) of the cake left.

- (b) Half an hour. There are several ways to do this calculation. For example, one third of three quarters is one quarter (in symbols, $\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$). Hence, two thirds (of three quarters) is two quarters, or a half: $\frac{2}{3} \times \frac{3}{4} = \frac{2}{4} = \frac{1}{2}$. Alternatively $\frac{3}{4} \times \frac{2}{3}$ gives $\frac{6}{12}$, i.e. $\frac{1}{2}$.

Another method is to cancel to give: $\frac{3^1}{4^2} \times \frac{2^1}{3^1} = \frac{1}{2}$. You might also have converted hours to minutes: $\frac{3}{4}$ now is 45 minutes, and $\frac{2}{3}$ of 45 minutes is $\frac{2}{3} \times 45 = \frac{90}{3} = 30$ minutes.

So check it after half an hour.

- 5 (a) $3 - 10 = -7$, so you would have $-£7$ left, i.e. you would have a £7 overdraft. Section 1.3.5
- (b) $-5 - 6 = -11$, so the temperature was -11°C .
- (c) (i) $-3 + -12 = -15$ (a debt of £3 plus a debt of £12 gives a debt of £15).
- (ii) $-4 - -11 = -4 + 11 = 7$ (incurring a debt of £4 and being let off a debt of £11 results in being £7 better off).
- (iii) $7 \times -6 = -42$ (7 debts of £6 result in a debt of £42).
- (iv) $-56 \div -7 = 8$ (a £56 debt divided into £7 debts gives eight £7 debts).

1.3.1 Arithmetic with whole numbers

When you are adding or subtracting whole numbers, an important thing to keep in mind is the place value of the figures. It is often a good idea to set out the numbers in columns before doing the arithmetic.

Example 10

- (a) There are 4985 people living in a village and 93 people living in a nearby hamlet. How many people live in the village and the hamlet together?
- (b) Of the total number of people in the village and hamlet, 1352 are under voting age. How many people are eligible to vote?

Solution

$$\begin{array}{r} (a) \quad 4985 \\ + \quad 93 \\ \hline 5078 \end{array}$$

There are 5078 people altogether.

$$\begin{array}{r} (b) \quad 5078 \\ - \quad 1352 \\ \hline 3726 \end{array}$$

3726 people are eligible to vote.

An alternative method is to note that 93 is 7 less than 100. So add 100 to 4985 (5085) and subtract 7 (5078).

To multiply and divide by 10, 100, 1000, etc., write the digits in their place value columns. To multiply, move the digits to the left (replacing the numbers on the right with zeros) and to divide move them to the right (putting in a decimal point, and any zeros necessary for the place value). Multiplication and division by whole numbers in general can be carried out by combining this technique with a knowledge of the multiplication tables up to 10.

Example 11

A machine in a factory puts tops on bottles at the rate of 1 every second.

- (a) How many bottles does it top in a 10 hour working day?
- (b) How many bottles does it top in a 30 day month, at 10 hours a day?

Solution

- (a) There are 60 seconds in a minute and so it puts on 60 tops a minute. There are 60 minutes in an hour so it puts on 60×60 tops in an hour, which gives 3600.

There are many ways to do this e.g. think of 60×60 as $6 \times 10 \times 6 \times 10$, which is $6 \times 6 \times 10 \times 10$, i.e. 36×100 , giving 3600 as before. However you tackle it, the important thing is to remember the place values of the numbers so it tops 3600 bottles an hour. In 10 hours it tops $3600 \times 10 = 36\,000$ bottles.

It tops 36 000 bottles in a 10 hour day.

- (b) To find out how many bottles it tops in 30 days, multiply 36 000 by 30. This is $36\,000 \times 3 \times 10 = 1\,080\,000$. So the factory machine tops 1 080 000 bottles a month (just over a million).

The factory puts on 1 080 000 bottle tops in 30 days.

Division is probably the most awkward of the four arithmetic operations. Since you have a calculator, you do not need to be able to carry out complicated divisions by hand, but you do need to carry out simple divisions in order to check your calculator calculations. Division is the reverse process of multiplication. The quantity $12 \div 3$ tells us how many times 3 goes into 12. Since $4 \times 3 = 12$, $12 \div 3 = 4$.

Example 12

Small candles are sold in boxes of 40. How many boxes do you need in order to have 1000 candles?

Solution

To solve the problem, you need to know how many times 40 goes into 1000. There are several ways to do this. One is to divide 1000 by 40: $\frac{1000}{40}$ is a fraction and so its value does not change if top and bottom are multiplied or divided by the same number. Here, notice that top and bottom are both divisible by 10.

$$\frac{1000 \div 10}{40 \div 10} = \frac{100}{4} = 25$$

So you need 25 boxes of candles.

Alternatively, you might say
1 box gives 40, so
10 boxes give 400,
20 boxes give 800,
5 boxes give 200,
so 25 boxes give 1000.

Try some yourself (1.3.1)

Solutions on page 97.

- 1 Carry out the following calculations, without using a calculator.
 - (a) a million pound lottery prize minus a three hundred pound administrative charge.
 - (b) The sum of cheques for £25 and £1029, to pay into a bank account.
 - (c) Divide a £27 000 jackpot prize equally among 9 people in the syndicate.
 - (d) Divide a £27 000 jackpot prize equally among 900 people in the syndicate.
- 2 Place the following times in order, shortest first:
four weeks; 29 days; 360 hours.
- 3 Describe, in your own words, the effect of multiplying any number by
(a) 1, (b) 0.

1.3.2 Order of calculations

You may have noticed that sometimes the order in which calculations are carried out seems to matter and sometimes it does not. When using a calculator, it is very important to know the order in which it will do calculations. It is not always the order in which you enter them.

Although written English is read from left to right, this is not the case for all written languages (Chinese is read top to bottom, right to left). With mathematics, the order of the written operations does not always indicate the order in which they should be carried out.

Example 13

- (a) Multiply 3 by 365. How many days are there in 3 years?
- (b) Divide 366 by 3 to find out how many days there are per term, in a 3-term leap year. What happens in other years?

Solution

- (a) Multiplying by 365 is quite hard work, whereas multiplying by 3 is comparatively easy. It makes no difference whether you multiply 3 by 365 or multiply 365 by 3.

$$3 \times 365 = 365 \times 3 = 1095$$

Since $3 \times 365 = 1095$, there are 1095 days in 3 years, provided one is not a leap year. If it were there would be 1096.

- (b) $366 \div 3 = 122$. So there will be 122 days. In other years one term would have to be 121 days.

3 lots of 365 are the same as 365 lots of 3.

Note that dividing 3 by 366 is *not* the same as dividing 366 by 3:

$\frac{366}{3}$ gives over a hundred, whereas $\frac{3}{366}$ is much less than 1.

When you multiply two numbers together, it makes no difference which number you write first.

When you divide one number by another, it *does* matter which number you write first.

If you add two numbers, the order does not matter,

$$3 + 2 = 2 + 3,$$

but the same is not true with subtraction,

$$3 - 2 \neq 2 - 3.$$

$\neq$ means 'is not equal to'.

When adding two numbers together, it makes no difference which number you write first.

When subtracting one number from another, it *does* matter which number you write first.

Sometimes you may want to make several calculations in succession, and the order in which the calculations are performed may or may not be significant. For example, if you want to add $12 + 7 + 13$, it makes no difference which of these two processes you adopt:

add the 12 and 7 first, to give 19, and then the 13, to give 32;

or

add the 7 and 13 first, to give 20, and then add this to 12 to give 32 again.

Brackets can be used in a calculation to mean 'do this first'. $(7 + 13) + 12$ is the same as $7 + (13 + 12)$. But sometimes the order of calculation *does* make a difference:

$$(7 - 12) + 13 \text{ is not the same as } 7 - (12 + 13).$$

Example 14

Calculate $(7 + 3) \times 2$ and $7 + (3 \times 2)$.

Solution

To find $(7 + 3) \times 2$, first do the calculation in brackets to get $7 + 3 = 10$, then multiply by 2 to get $10 \times 2 = 20$. For the second calculation, to find $7 + (3 \times 2)$, first calculate 3×2 to give 6, then add to 7 to give 13. So $(7 + 3) \times 2 \neq 7 + (3 \times 2)$.

Brackets are not always given in a calculation, and there are rules which tell you in which order to do the calculations in the absence of brackets. The following is the order that the course calculator will use.

Calculations are performed in the following order:

Brackets;

Powers (e.g. squaring or cubing a number);

Division and Multiplication (performed in the order written, left to right);

Addition and Subtraction (performed in the order written, left to right).

If you have a calculator handy, check that it follows these rules.

Example 15

Calculate $10 + 3 \times 7$.

Solution

Multiplication is done before addition (unless brackets tell you otherwise) so

$$10 + 3 \times 7 = 10 + 21 = 31.$$

It is said that multiplication 'takes precedence' over addition.

Sometimes brackets are implied in division. For example,

$$\frac{13 + 14}{3 + 6} = \frac{(13 + 14)}{(3 + 6)} = (13 + 14) \div (3 + 6) = 27 \div 9 = 3.$$

Try some yourself (1.3.2)

Solutions on page 97.

- 1 Look at the rules in the boxes on page 24.

Write in your own words the rules for multiplying and dividing, adding and subtracting two numbers, giving an example of each.

- 2 Carry out the following calculations, without your calculator.

(a) $3 \times (60 + 70)$ (b) $(3 \times 60) + 70$

(c) $(70 - 60) \div 5$ (d) $70 + (210 \div 30)$

Check your answers using your calculator.

- 3 Insert brackets in the following calculations to emphasize the order in which the course calculator would perform them, then do the calculations by hand and on your calculator, with and without the brackets, as a check.

(a) $3 \times 60 + 70$ (b) $10 - 15 \div 5$ (c) $20 - 2 \times 8$

(d) $3 + 16 - 10$ (e) $3 \times 10 \div 5$

- 4 In which of the calculations in Question 3 does the order of the calculations make a difference?



- 5 (a) Calculate $\frac{20 + 4}{3 + 5}$.

(b) Does the calculation in part (a) represent an exception to the 'division before addition' rule?

1.3.3 Arithmetic with decimals

Arithmetic with decimals is much the same as arithmetic with whole numbers, but continue to take care with place value.

Addition and subtraction can be carried out with decimals just as for whole numbers. Take care to write the numbers in their correct columns and keep the decimal points in line under each other.

Example 16

- (a) Add 303.035 kg and 7.77 kg.
(b) Subtract 7.77 kg from 303.035 kg.

Solution

$$\begin{array}{r} \text{(a)} \quad 303.035 \\ + \quad 7.770 \\ \hline 310.805 \end{array}$$

So the answer is 310.805 kg.

$$\begin{array}{r} \text{(b)} \quad 303.035 \\ - \quad 7.770 \\ \hline 295.265 \end{array}$$

So the answer is 295.265 kg.

Remember to supply a '0' at the end of 7.77 to keep the number of decimal places the same.

With an awkward subtraction like this, a useful check is to add 7.77 to 295.265 to make sure that you get back to 303.035.

Multiplication and division of decimal numbers are carried out just as with whole numbers, except that now you can carry out the division even when one number does not divide exactly into the other. You also need to take care about the position of the decimal point. So check your answer is sensible.

Example 17

- (a) If somebody's hair grows at the rate of 0.4 cm a week, how much will it grow in 52 weeks or one year?
(b) How long would it take to grow their hair 10 cm?
(c) How long would it take to grow their hair 3 cm?

Solution

- (a) In 52 weeks it grows 52×0.4 cm. To multiply by 0.4, multiply by 4 and shift the digits one place to the right, $52 \times 4 = 208$. So

$$52 \times 0.4 = 20.8.$$

So the hair will grow 20.8 cm in a year. (0.4 is just under a half and half of 52 is 26, so the answer is sensible.)

- (b) You need to know how many times 0.4 cm goes into 10 cm, so divide 10 by 0.4,

$$10 \div 0.4 = \frac{10}{0.4} = \frac{100}{4} \quad (\text{multiply top and bottom by 10}).$$

Now carry out the division: $100 \div 4 = 25$. So it would take 25 weeks to grow 10 cm. (Check: 10 cm is about half of the previous answer of 20.8 cm which took a year, so will take about half a year. 25 weeks is near enough half a year, so the answer is sensible.)

Notice that multiplying by 0.4 gives an answer smaller than 52.

- (c) There are several ways of doing this. You might do it formally as

$$3 \div 0.4 = \frac{3}{0.4} = \frac{3 \times 10}{0.4 \times 10} = \frac{30}{4}.$$

Carrying out the division gives 7.5.

Alternatively you might do it more informally: it grows 4 cm in 10 weeks, and 3 cm is three-quarters of 4 cm, so the answer is three-quarters of 10 weeks, i.e. $\frac{3}{4} \times 10 = 7.5$ weeks.

Therefore it would take 7.5 weeks to grow 3 cm. (Check: 3 cm is a bit less than a third of 10 cm which takes 25 weeks, so will take a bit less than a third of 25 weeks. 7.5 weeks is near enough a third of 25 weeks, so the answer is sensible.)

The same rules about the order of calculations apply to decimals as apply to whole numbers.

Example 18

Calculate $3.5 + 0.7 \times 8 \div (4.6 - 2.6)$.

Solution

Brackets first: $4.6 - 2.6 = 2.0$, so calculate $3.5 + 0.7 \times 8 \div 2.0$.

Multiply next: $0.7 \times 8 = 5.6$, so calculate $3.5 + 5.6 \div 2.0$.

Divide next: $5.6 \div 2 = 2.8$, so calculate $3.5 + 2.8$.

Lastly add: $3.5 + 2.8 = 6.3$.

So the answer is 6.3.

Try some yourself (1.3.3)

Solutions on page 98.

- Without using your calculator, find the following.
 - $100.001 + 10.1$
 - $100.001 - 10.1$
 - $75.6 \div 0.6$
 - 75.6×0.6
 - $100.001 + 75.6 \div 0.6$
 - $(100.001 + 10.1) \times 60$
- Try dividing 10 by 3, first without using your calculator, and giving your answer as a decimal. What is the difficulty? What answer does the calculator give?



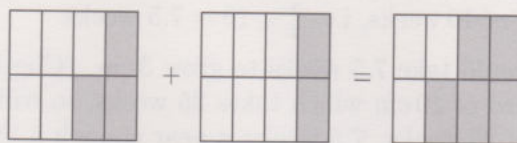
1.3.4 Arithmetic with fractions

Addition and subtraction with fractions can be quite awkward, particularly when the fractions are mixed with whole numbers and when the fractions have different denominators. In such cases you may need to use your calculator. However simple fractions can be done by hand, and it is useful to gain confidence in handling simple fractions.

To add two fractions with the same denominator, just add the numerators. For example,

$$\frac{3}{8} + \frac{2}{8} = \frac{3+2}{8} = \frac{5}{8}.$$

Just as 3 eggs plus 2 eggs is 5 eggs, so 3 eighths plus 2 eighths is 5 eighths.



To subtract one fraction from another, where the denominators are the same, just subtract one numerator from the other. For example,

$$\frac{5}{12} - \frac{3}{12} = \frac{2}{12}.$$

Just as 5 oranges minus 3 oranges is 2 oranges, so 5 twelfths minus 3 twelfths is 2 twelfths, but $\frac{2}{12}$ is the same as $\frac{1}{6}$. So the simplest answer is $\frac{1}{6}$.



A useful mathematical technique with things you can't immediately handle is to put them in a form where you *can* handle them.

However, if the denominators are different, the calculation is a bit more awkward. The first thing to do is to rewrite the fractions as equivalent fractions where the denominators are the same.

Example 19

- (a) Evaluate $\frac{1}{4} + \frac{3}{8}$.
 (b) Write $2\frac{1}{3}$ as a simple fraction.

Solution

- (a) You cannot add quarters and eighths directly, just as you cannot add queens and eggs. You need to change them into the same thing. It is difficult to change queens to eggs (unless you are a magician), but possible to change quarters to eighths. So use the fact that $\frac{1}{4} = \frac{2}{8}$ to make the denominators the same. Putting this into the calculation gives

$$\frac{1}{4} + \frac{3}{8} = \frac{2}{8} + \frac{3}{8} = \frac{5}{8}.$$

- (b) $2\frac{1}{3} = 2 + \frac{1}{3}$. 2 is the same as $\frac{2}{1} = \frac{2 \times 3}{1 \times 3} = \frac{6}{3}$. So $2\frac{1}{3} = \frac{6}{3} + \frac{1}{3} = \frac{7}{3}$.

Example 20

Evaluate $\frac{1}{2} + \frac{1}{3}$.

$2\frac{1}{3}$ is equivalent to $\frac{7}{3}$.

To enter $2\frac{1}{3}$ on a calculator, you can input $2 + \frac{1}{3}$. $2\frac{1}{3}$ is called a 'mixed number'.

Solution

You cannot change halves to thirds or vice versa, but you can change them both to sixths.

$$\frac{1}{2} = \frac{3}{6} \text{ and } \frac{1}{3} = \frac{2}{6}. \text{ So } \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}.$$

To add or subtract fractions, use equivalent fractions to make the denominators the same and then add or subtract the numerators.

Multiplication by a whole number is really just repeated addition, 4×3 is 4 times 3:

$$4 \times 3 = \underbrace{3 + 3 + 3 + 3}_{4 \text{ times}}.$$

In the same way,

$$4 \times \frac{3}{5} = \frac{3}{5} + \frac{3}{5} + \frac{3}{5} + \frac{3}{5} = \frac{12}{5}.$$

You may have noticed that the answer is the same as if you had multiplied the numerator of $\frac{3}{5}$ by 4:

$$4 \times \frac{3}{5} = \frac{4 \times 3}{5} = \frac{12}{5}.$$

Now look at an example where two fractions are multiplied.

This is the same as 3 times 4.

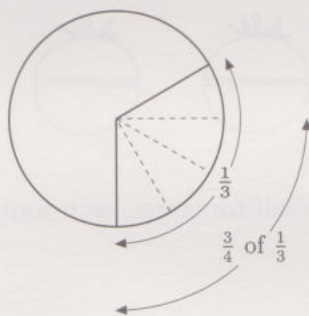
Just add the numerators since the denominators are the same.

Example 21

Isaac and Jasmine are helping themselves to some pizza. Isaac takes $\frac{1}{3}$ of the pizza and gives $\frac{3}{4}$ of this portion to Jasmine. How much pizza does Jasmine get?

Solution

She gets $\frac{3}{4}$ of $\frac{1}{3}$, which just means $\frac{3}{4} \times \frac{1}{3}$.



Drawing a diagram can be a good way of seeing what is going on.

Isaac's portion is $\frac{1}{3} = \frac{4}{12}$ of the pizza. Divide the $\frac{1}{3}$ (i.e. $\frac{4}{12}$) into 4 portions and Jasmine gets 3 of them. Each piece is $\frac{1}{12}$ of the whole pizza, and three of these pieces constitute $\frac{3}{12}$ of the whole pizza. So the answer is that Jasmine gets three twelfths of the pizza. The calculation to which this corresponds is

$$\frac{3}{4} \times \frac{1}{3} = \frac{3}{12}.$$

Notice that $\frac{3}{12} = \frac{1}{4}$, so an equivalent answer is that Jasmine gets a quarter of the pizza.

Notice that in the example above, when multiplying two fractions together you could just multiply the numerators and multiply the denominators, i.e.

$$\frac{3}{4} \times \frac{1}{3} = \frac{3 \times 1}{4 \times 3} = \frac{3}{12}.$$

This is true in general.

To give the answer in its simplest form, divide top and bottom by 3 to give

$$\frac{3 \div 3}{12 \div 3} = \frac{1}{4}.$$

You can confirm from the diagram above that Jasmine's share is a quarter of the whole pizza. There is a quicker way of getting to the answer in its simplest form, known as *cancelling*. Look again at the fraction

$$\frac{3 \times 1}{4 \times 3}.$$

Divide top and bottom by 3, and show this by crossing off, or cancelling, the threes,

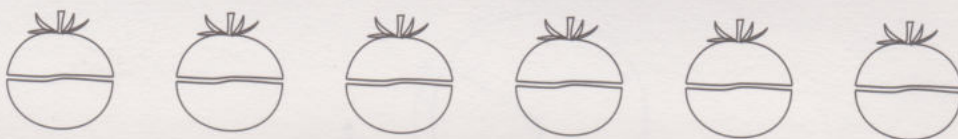
$$\frac{\cancel{3} \times 1}{4 \times \cancel{3}} = \frac{1}{4}.$$

To multiply fractions, multiply the denominators together and the numerators together, cancelling any factors which the numerators and denominators have in common.

Before considering division of fractions, it is helpful to think about division of whole numbers.

$6 \div 2$ asks for the number of *twos* in 6: $6 \div 2 = 3$, since *three* twos are six ($3 \times 2 = 6$).

In a similar way, $6 \div \frac{1}{2}$ is asking for the number of *halves* in 6. Suppose a friend is making salad decorations for plates of sandwiches. He has 6 tomatoes and wants to know how many half tomatoes this will give. This will be $6 \div \frac{1}{2}$. So think of the 6 as 6 whole tomatoes.

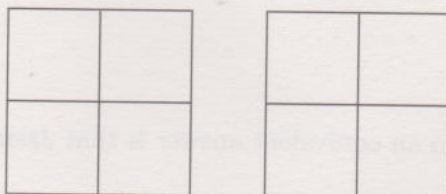


Each tomato contains two half-tomatoes, so 6 tomatoes contain 6×2 half-tomatoes. Hence

$$6 \div \frac{1}{2} = 6 \times 2 = 12.$$

Thus dividing by $\frac{1}{2}$ is the same as multiplying by 2.

Similarly $2 \div \frac{1}{4}$ is asking for the number of quarters in 2. Think of two cakes.



Each cake contains 4 quarters: the two cakes contain 8 quarters.

Expressed in figures this is

$$2 \div \frac{1}{4} = 2 \times 4 = 8.$$

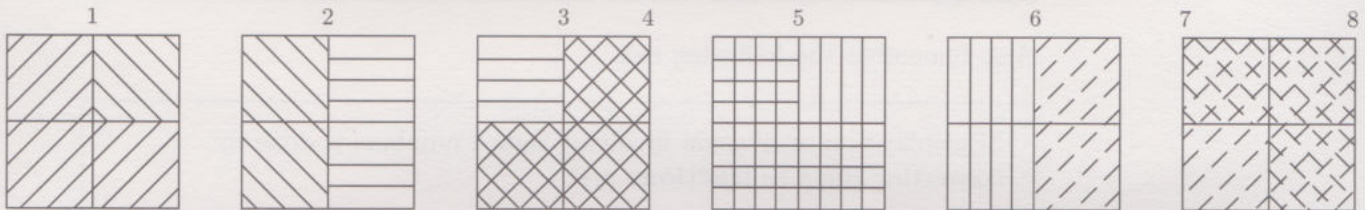
Hence dividing by $\frac{1}{4}$ is the same as multiplying by 4.

This illustrates the rule for dividing by fractions of the form $\frac{1}{n}$.

To divide by $\frac{1}{n}$, multiply by n (n can be any number except zero).

How about other fractions? What is $6 \div \frac{3}{4}$? Three quarters is $3 \times \frac{1}{4}$. So dividing by $\frac{3}{4}$ is the same as dividing by 3 and by $\frac{1}{4}$. Dividing by $\frac{1}{4}$ is multiplying by 4. So dividing by $\frac{3}{4}$ is the same as dividing by 3 and, multiplying by 4, which is multiplying by $\frac{4}{3}$. So $6 \div \frac{3}{4} = 6 \times \frac{4}{3} = \frac{24}{3} = 8$.

To see whether this answer is sensible, consider six squares, divided into quarters, and count how many groups of three quarters there are.



The general rule for dividing by a fraction can be stated as follows.

To divide by a fraction, turn the fraction upside down and multiply.

Does this also work for dividing by $\frac{1}{n}$? Yes, because multiplying by n is the same as multiplying by $\frac{n}{1}$. It also works for division of whole numbers: dividing by 4 ($= \frac{4}{1}$) is the same as multiplying by $\frac{1}{4}$.

Example 22

Evaluate (a) $\frac{1}{3} \div \frac{5}{6}$, (b) $1\frac{1}{2} \div 3$.

Solution

(a) Using the rule above,

$$\frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \times \frac{6}{5}.$$

This is

$$\frac{6}{3 \times 5} = \frac{6}{15} = \frac{2}{5} \text{ (divide top and bottom by 3).}$$

Alternatively divide top and bottom by 3 by cancelling by 3 to give

$$\frac{1 \times \cancel{6}^2}{\cancel{18}^3 \times 5} = \frac{2}{5}.$$

So $\frac{1}{3} \div \frac{5}{6} = \frac{2}{5}.$

(b) $1\frac{1}{2}$ is $1 + \frac{1}{2} = \frac{1}{1} + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}.$

So $1\frac{1}{2} \div 3 = \frac{3}{2} \div \frac{3}{1} = \frac{3}{2} \times \frac{1}{3} = \frac{\cancel{3} \times 1}{2 \times \cancel{3}} = \frac{1}{2}.$

In multiplication or division involving mixed numbers like $1\frac{1}{2}$, convert the mixed number to a (top heavy) fraction.

Turning a fraction upside down is called finding its **reciprocal**.

The reciprocal of $\frac{5}{6}$ is $\frac{6}{5}$. The reciprocal of $\frac{1}{6}$ is $\frac{6}{1}$, i.e. 6.

So an alternative formulation of the rule for dividing by a number can be stated as follows.

Division by a number is the same as multiplication by its reciprocal.

Also remember the following rule.

Multiplication or division involving mixed numbers is done by converting them to **fractions** first.

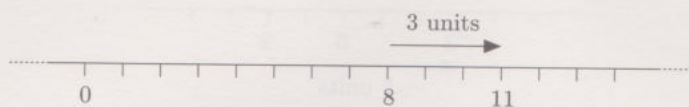
Solutions on page 98.

Try some yourself (1.3.4)

- 1 Evaluate each of the following.
 (a) $\frac{3}{10} + \frac{5}{10}$ (b) $\frac{11}{32} - \frac{7}{32}$ (c) $\frac{7}{16} - \frac{2}{16}$ (d) $\frac{1}{3} + \frac{1}{6}$ (e) $\frac{1}{3} + \frac{1}{5}$
- 2 Explain in your own words how to add two fractions, as if to someone who does not know how to do it.
- 3 Evaluate each of the following.
 (a) $\frac{11}{12} \times \frac{3}{11}$ (b) $\frac{5}{6} \times \frac{9}{10}$ (c) $\frac{3}{4}$ of 2 (d) $\frac{4}{5}$ of $\frac{1}{2}$
- 4 A school contains 720 pupils. $\frac{5}{8}$ are boys and $\frac{3}{8}$ are girls. How many girls are there in the school?
- 5 Evaluate each of the following, giving each answer in its simplest form.
 (a) $10 \div \frac{1}{2}$ (b) $\frac{2}{5} \div \frac{5}{7}$ (c) $1\frac{1}{3} \div 2$
- 6 Explain how to multiply fractions, to somebody who does not know how to do it.

1.3.5 Arithmetic with negative numbers

In order to understand arithmetic with negative numbers, it is helpful to see how arithmetic can be represented on the number line. The strategy is to start with simple examples of whole positive numbers and then generalize to negative numbers. The same principles must apply! For example, to find $8 + 3$ start at 8 and move 3 units to the right.



giving $8 + 3 = 11$. Adding a positive number means moving to the right along the number line.

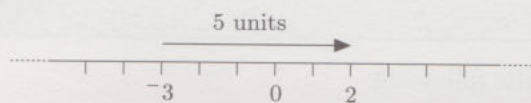
Another way of considering the arithmetic of positive and negative numbers is to consider them as the total value of the contents of a piggy bank, belonging to a child (Thomas). The numbers on the above number line can represent the value of Thomas's piggy bank. Calculations represent transactions involving the piggy bank. When Thomas gets pocket money or gifts of money, he *adds* them into his piggy bank. If Thomas had £8 in his piggy bank and adds £3 (from his pocket money) the value of the piggy bank in pounds is $8 + 3 = 11$.

When he spends money (on toys usually) he takes money out (*subtracts*). If there is not enough money in the piggy bank, he needs to borrow money. When he borrows money from his family, they note the debt on a piece of paper headed 'IOU' (I owe you). Thomas puts the 'IOUs' into his piggy bank. IOUs represent a negative amount of money (or debts).

Now suppose, at another time, Thomas's piggy bank contains an IOU for £3 when Thomas receives a gift of £5 to add to his piggy bank. £3 of this pays off the IOU and he has £2 left in the piggy bank. The transaction is represented by the calculation of the new value (in pounds) of the piggy bank

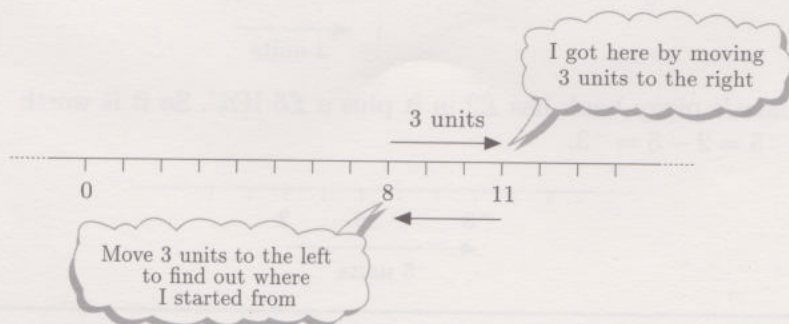
$$-3 + 5 = 2.$$

How is this represented on the number line? Adding a positive number is moving to the right.



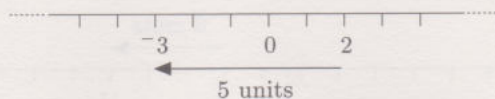
Moving 5 units to the right from -3 gets us to 2. So $-3 + 5 = 2$.

Now what about subtraction? You can think of subtraction as undoing addition: adding 3 to 8 gets you 11, and so subtracting 3 from the answer, 11, gets you back to 8. Therefore, in terms of the number line, subtracting 3 from 11 means starting at 11 and moving 3 units to the left.



So $11 - 3 = 8$. Subtracting a positive number means moving to the left along the number line. Thinking in terms of the contents of the piggy bank: if Thomas had £11 in his piggy bank and subtracts £3 (to buy a toy), he is left with £8.

Suppose on one occasion Thomas has £2 in his piggy bank. He wants to buy a toy for £5. $2 - 5$ means starting at 2 and moving 5 units to the left.



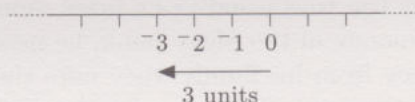
This takes you back to -3, so $2 - 5 = -3$.

So Thomas needs to borrow £3 and gets an IOU (from his mother) which he puts in his piggy bank. His piggy bank is worth -£3.

Now think about adding a negative number, by looking at the number line again. Suppose you start at 0. Since $0 + -3$ is the same as -3, you would expect that adding -3 to 0 on the number line should take you to the point marked -3 on the number line.

$$0 + -3 = -3$$

Suppose Thomas's piggy bank were empty and he added an IOU for £3 to it. The value of the piggy bank would be -3.



To reach -3 from 0, move 3 units to the left. In general to add a negative number you move to the left along the number line.

Adding a negative number is the same as subtracting the corresponding positive number.

In terms of Thomas's piggy bank, a negative number is an IOU. Adding an IOU is the same as taking money out (subtracting a positive number).

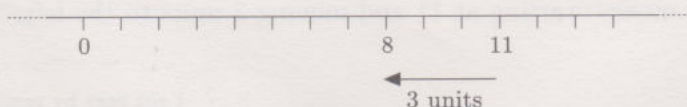
Example 23

Evaluate each of the following:

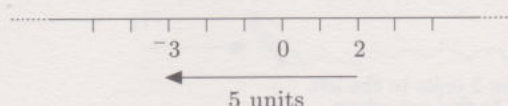
- (a) $11 + -3$ (b) $2 + -5$

Solution

- (a) In piggy bank terms, this represents contents of £11 plus a £3 IOU. So the overall value is $11 + -3 = 11 - 3 = 8$.

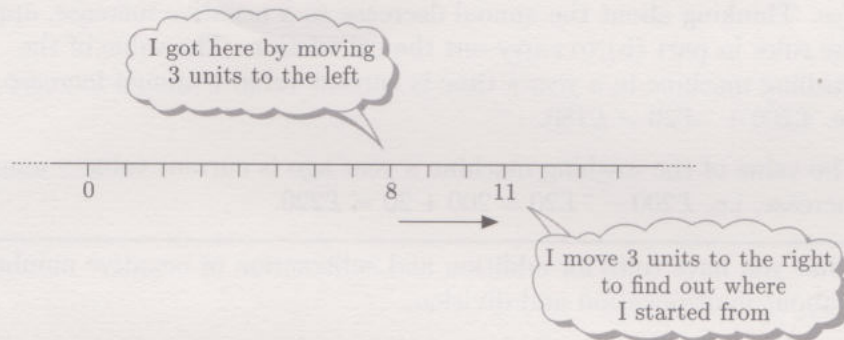


- (b) Thomas's piggy bank has £2 in it plus a £5 IOU. So it is worth $2 + -5 = 2 - 5 = -3$.



Next consider subtraction of a negative number. In terms of Thomas' piggy bank, subtracting a negative number is the same as taking away one of his IOUs. If his mother says 'you have been a good boy today so I'll take away that IOU for £3' this is equivalent to him being given £3. So $-(-3) = 3$. Does this correspond with the number line interpretation of subtracting a negative number?

Consider the evaluation of $8 - -3$. Continue to think of subtraction as undoing addition: adding -3 to 11 gives 8, and so *subtracting* -3 from the answer, 8, gets back to 11. So, in terms of the number line, subtracting -3 from 8 means starting at 8 and moving 3 units to the right.



But moving 3 units to the right is the same as adding 3. So

$$8 - -3 = 8 + 3 = 11.$$

If on one occasion Thomas's piggy bank is worth £8 in total (cash and IOUs) and his mother takes away a £3 IOU, the value of the piggy bank is $8 - -3 = 8 + 3 = 11$.

Subtracting a negative number is the same as adding the corresponding positive number.

Example 24

- If the value of a painting increases by £20 a year and it is worth £200 today, how much will it be worth in a year's time? How much was it worth a year ago?
- Describe in words how to calculate the value of an object like a picture one year in the future or one year ago, given a constant annual increase.
- If the value of a washing machine decreases by £20 a year and it is worth £200 today, how much will it be worth in a year's time? How much was it worth a year ago?
- If you regard a decrease as a negative increase, does your answer to (b) apply to the washing machine in (c)?

Solution

- (a) The value of the painting in a year's time is $\pounds 200 + \pounds 20 = \pounds 220$. The value of the painting a year ago was $\pounds 200 - \pounds 20 = \pounds 180$.
- (b) To work out the value a year in the future, add the annual increase to the current value. To work out the value a year in the past, subtract the annual increase from the current value.
- (c) The value of the washing machine in a year's time is $\pounds 200 + \pounds 20 = \pounds 220$. The value of the washing machine a year ago was $\pounds 200 - \pounds 20 = \pounds 180$.

Adding a negative increase is the same as subtracting the decrease. Subtracting a negative increase is the same as adding the decrease.

- (d) Yes. Thinking about the annual decrease as a negative increase, apply the rules in part (b) to carry out the calculation. The value of the washing machine in a year's time is current value + annual increase, i.e. $\pounds 200 + \pounds 20 = \pounds 220$.

The value of the washing machine a year ago is current value - annual increase, i.e. $\pounds 200 - \pounds 20 = \pounds 180$.

Now that you have rules for addition and subtraction of negative numbers, think about multiplication and division.

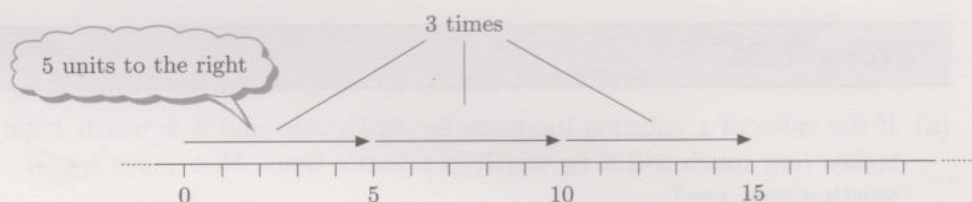
Example 25

Describe each of the following in terms of the number line and the value of Thomas's piggy bank:

- (a) the multiplication of 3 times 5 (that is, 3×5);
- (b) the multiplication of 3 times -5 (that is, 3×-5).

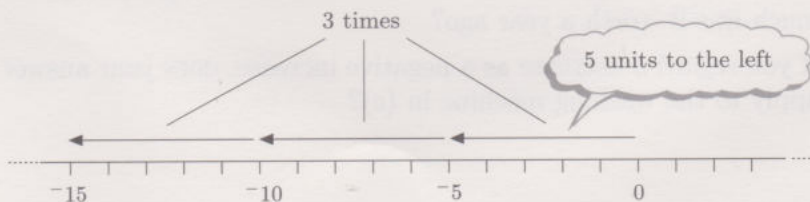
Solution

- (a) Three times 5 means adding three lots of 5, or do the move '5 units to the right' three times.



Thomas's piggy bank has three £5 notes in it. The value of the piggy bank in pounds is $3 \times 5 = 15$.

- (b) 3 times -5 means adding three lots of -5, or do 3 times 'move 5 units to the left'.



Thomas's piggy bank has 3 IOUs each for £5 in it. The value of the piggy bank in pounds is $3 \times -5 = -15$.

So multiplying a negative number by a positive number gives a negative answer. But what does it mean to multiply *by* a negative number—how can you add something a negative number of times?

Think of multiplication by a positive number as repeated addition, and of multiplication by a negative number as repeated subtraction. If Thomas' mother says she'll take away three of his £5 IOUs, it is equivalent to giving him three £5 notes. So $-3 \times -5 = 3 \times 5 = 15$. He is £15 better off.

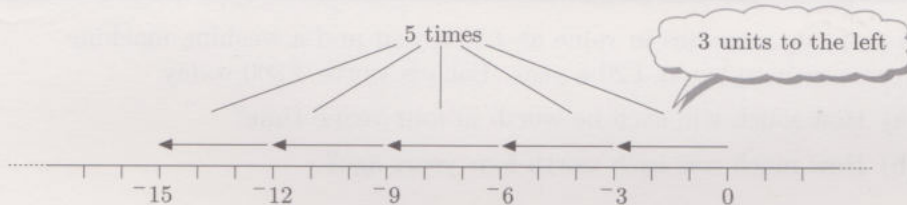
Example 26

Evaluate each of the following in terms of the number line and the value of Thomas' piggy bank.

- (a) 5×-3 (b) -5×-3

Solution

- (a) Think of 5 times -3 as meaning '5 lots of -3 '. So 5×-3 means: do 5 times 'move 3 units to the left'.

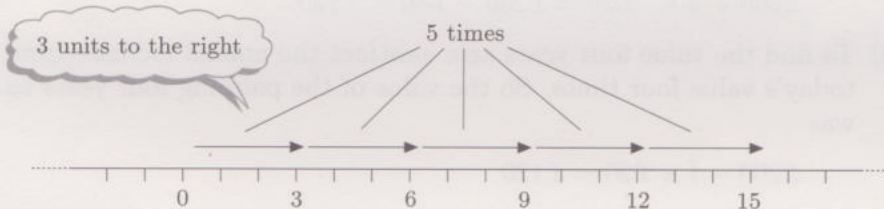


So $5 \times -3 = -15$.

In terms of the piggy bank Thomas has five £3 IOUs in it. The value in pounds is $5 \times -3 = -15$, which is equivalent to one £15 IOU.

- (b) Thinking in terms of repeated subtraction, do 5 times 'subtract -3 '. Subtracting -3 means adding 3, or moving 3 units to the right. So do 5 times 'move 3 units to the right'.

Subtracting -3 is the same as adding 3.



So $-5 \times -3 = 5 \times 3 = 15$.

Thomas's mother takes away five £3 IOUs. This is equivalent to giving him £15. The calculation in pounds is $-5 \times -3 = 5 \times 3 = 15$.

The rules for multiplying positive and negative numbers can be illustrated by the table in the margin.

$\times$	+	-
+	+	-
-	-	+

Multiplying a positive number by a positive number gives a positive answer.

Multiplying a negative number by a positive number gives a negative answer.

Multiplying a positive number by a negative number gives a negative answer.

Multiplying a negative number by a negative number gives a positive answer.

The painting and the washing machine from Example 24 can be used again, to illustrate these rules.

Example 27

A painting increases in value at £20 a year and a washing machine decreases in value at £20 a year. Each is worth £200 today.

- How much will each be worth in four years' time?
- How much was each worth four years ago?

Solution

- To find the value in four years' time, in each case add the annual increase (which is negative in the case of the washing machine) four times. So the value of the painting will be

$$£200 + 4 \times £20 = £200 + £80 = £280$$

and the value of the washing machine will be

$$£200 + 4 \times -£20 = £200 - £80 = £120.$$

- To find the value four years ago, subtract the annual increase from today's value four times. So the value of the painting four years ago was

$$£200 - 4 \times £20 = £120$$

and the value of the washing machine four years ago was

$$£200 - 4 \times -£20.$$

But this is a matter of repeatedly subtracting a negative number, which is the same as repeatedly adding the corresponding positive number. So

$$£200 - 4 \times -£20 = £200 + 4 \times £20 = £280.$$

Lastly consider division. Dividing 8 by 2 means 'How many times does 2 go into 8?' or 'What must you multiply 2 by to get 8?'. The answer is 4.

So to find $8 \div -2$, you need to ask 'What do I have to multiply -2 by to get 8?'. The answer is -4, since $-2 \times -4 = 8$. So $8 \div -2 = -4$.

Similarly, to find $-8 \div -2$ you need to ask 'what do I have to multiply -2 by to get -8?' and the answer is 4, since $-2 \times 4 = -8$. So $-8 \div -2 = 4$.

Using this sort of argument you can work out the rules for division of and

by negative numbers. They are remarkably similar to those for multiplication (as you might expect, since they are inverse processes).

Multiplying or dividing a positive number by a positive number gives a positive answer.

Multiplying or dividing a negative number by a positive number gives a negative answer.

Multiplying or dividing a positive number by a negative number gives a negative answer.

Multiplying or dividing a negative number by a negative number gives a positive answer.

$\div$	+	-
+	+	-
-	-	+

The last one is the most difficult to remember. If you are in doubt, you can always use your calculator to check.

These rules may be summarized as follows.

Multiplying or dividing two numbers of the same sign gives a positive answer.

Multiplying or dividing two numbers of different signs gives a negative answer.

Try some yourself (1.3.5)

Solutions on page 99.

- Evaluate each of the following and give an example from everyday life to illustrate the sum (e.g. Thomas's piggy bank).
 (a) $-4 - 6$ (b) $3 + -5$ (c) $-4 + -7$ (d) $-5 + 8$
 (e) $4 - -2$ (f) $-3 - -5$
- Kim was walking in Israel. She started at 37 metres below sea level and ended up at 42 metres above sea level. How far had she climbed up?
- Give examples of all the rules for the division of and by positive and negative numbers.
- Evaluate each of the following and where possible give an example from everyday life to illustrate the calculation.
 (a) 4×-3 (b) -2×-7 (c) -3×5
 (d) $24 \div -4$ (e) $-40 \div -4$ (f) $-45 \div 15$

If you have a calculator check that it gives the same answers as yours.

Outcomes

Now that you have studied Module 1 you should be able to:

- ◇ write whole numbers and decimals in place-value columns and compare their sizes;
- ◇ multiply and divide whole numbers and decimals by 10, 100, 1000 and so on;
- ◇ indicate given fractions on a diagram and find equivalent fractions for a given fraction;
- ◇ mark numbers on a number line;
- ◇ choose appropriate units for a given purpose;
- ◇ convert between different metric units and different time units;
- ◇ perform, without a calculator, calculations involving: brackets, addition, subtraction, multiplication and division of whole numbers, decimals, simple fractions and negative numbers;
- ◇ describe in words the arithmetic processes that you use.

Module 2 Rounding and estimation

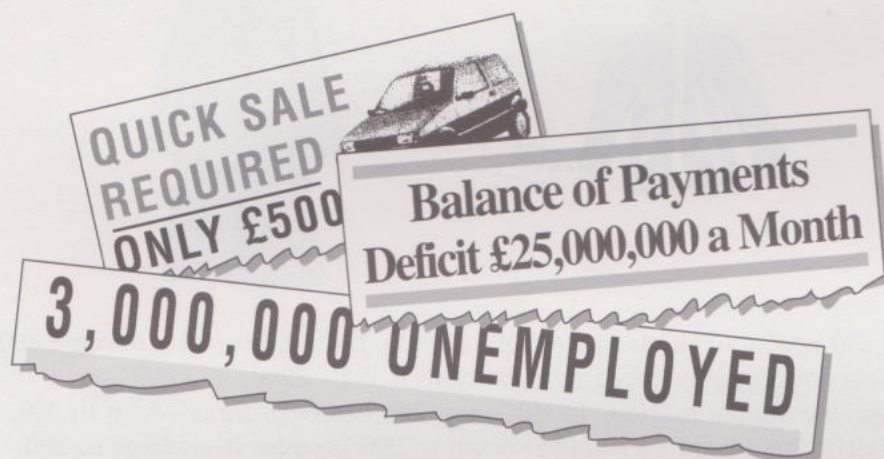
For many of the calculations you will carry out in MU120, you will be using the course calculator. The main aim of this module is to help you to do this in a sensible and fruitful way. Using a calculation to solve a problem involves four main stages:

- Stage 1 working out what calculation you want to do;
- Stage 2 working out roughly what size of answer to expect from your calculation;
- Stage 3 carrying out the calculation;
- Stage 4 interpreting the answer: does it agree with the rough estimate?
Does it make sense in terms of the original problem?

This module focuses on Stages 2 and 4, in preparation for using your calculator.

2.1 Rounding

Newspapers, magazines and television often provide numerical information such as that below.



How do you interpret the figures in the headlines above?

It is unlikely that the monthly balance of payments deficit, for example, would be exactly £25 000 000. It's more likely to be a number fairly close to £25 000 000: a number like £24 695 481 or £25 332 206. From the reader's point of view, £25 000 000 gives an idea of the size of the deficit; it's a *good approximation*. Large numbers in particular are often approximated in this way. This process is called **rounding**.

When you use your course calculator to do a calculation, you will need to interpret the results. Part of this interpretation is rounding the answer appropriately.

Remember 25,000,000 means the same as 25 000 000.

Try these first

- 1 The population of a village is 5481. Round this:
 - (a) to the nearest thousand people;
 - (b) to the nearest hundred people.

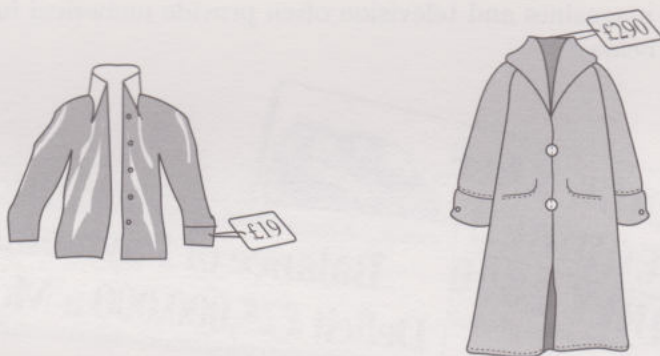
These are diagnostic questions. Try them to see which topics you need to revise.

- 2 Round a measurement of 1.059 metres:
- to the nearest whole number of metres;
 - to two decimal places;
 - to two significant figures.

Check your answers

- Follow-up in Section 2.1.1 1 (a) 5481 rounded to the nearest thousand people is 5000 people.
 (b) 5481 rounded to the nearest hundred people is 5500 people.
- Follow-up in Section 2.1.2 2 (a) 1.059 rounded to the nearest whole number of metres is 1 metre.
 (b) 1.059 rounded to two decimal places is 1.06 m.
- Follow-up in Section 2.1.3 (c) 1.059 rounded to two significant figures is 1.1 m.

2.1.1 Rounding whole numbers



You will probably think to yourself that the coat shown costs about £300. £290 is considerably closer to £300 than it is to £200, so £300 is a reasonable approximation. In this case, 290 has been rounded up to 300. Similarly, 208 would be rounded down to 200 because it is closer to 200 than it is to 300. Both numbers have been *rounded to the nearest hundred pounds*.

When *rounding to the nearest hundred*, anything below fifty rounds down. So 248 rounds to 200. Anything over fifty rounds up. So 253 rounds to 300. Technically there is a problem with fifty itself as it is equidistant from the hundred above and the hundred below. Usually fifty is rounded up, so that a simple rule can be applied:

When *rounding to the nearest hundred* look at the tens digit:

- if the tens digit is 0, 1, 2, 3, 4 round down;
- if the tens digit is 5, 6, 7, 8, 9 round up.

So £284 rounds up (to £300) and £233 rounds down (to £200).

The blouse in the figure was £19 and you may well have thought of it as roughly £20. In this case you would be *rounding to the nearest ten* (pounds).

The rule for rounding to the nearest hundred can be adapted easily to rounding to the nearest ten. Instead of looking at the tens digit look at the units digit.

So £23 is rounded down (to £20) and £36 is rounded up (to £40).

Digits are explained in Module 1, Section 1.1.1.

Numbers are often approximated to make them easier to handle, but sometimes it doesn't help very much to round to the nearest 10 or the nearest 100 if the number is very large. For example, suppose the monthly balance of payments deficit was actually £24 695 481. Rounded to the nearest 10, it's £24 695 480; and to the nearest 100, it's £24 695 500. But £24 695 500 is still a complicated number to deal with in your head. That's why it was rounded to £25 000 000 in the newspaper headline. In fact numbers can be rounded to any level of accuracy—to the nearest 1000, or the nearest 10 000 or even to the nearest million (as in the newspaper headline).

**Balance of Payments
Deficit £25,000,000 a Month**

The rule for rounding can be generalized as follows.

Rounding to the nearest....

Look at the digit to the right of the one you are rounding to:

- if the digit is 0, 1, 2, 3, 4 round down;
- if the digit is 5, 6, 7, 8, 9 round up.

Notice that this general rule checks with rounding to the nearest hundred (the digit to the right of the hundreds digit is the tens digit) and rounding to the nearest ten (the digit to the right of the tens digit is the units digit).

Thus, £24 695 481 rounded to the nearest 1000 is £24 695 000 and rounded to the nearest million, or 1000 000 is £25 000 000.

Example 1

The distance from London to Newcastle is 282 miles. Round this distance to the nearest 10 miles and the nearest hundred miles.

Solution

To round to the nearest 10, look at the units digit. This is a 2, so round down to 280 miles.

To round to the nearest 100, look at the tens digit. This is 8, so round up to 300 miles.

Therefore the distance from London to Newcastle is 280 miles, to the nearest ten miles, and 300 miles to the nearest hundred miles.

Solutions on page 99.

Try some yourself (2.1.1)

1 Complete the following table

Number	Rounded to nearest		
	10	100	1000
325 089			
45 982			
11 985			

2 Some of the entries in the following table are incorrect. Identify the errors and correct them.

Number	Rounded to nearest		
	10	100	1000
254 987	254 980	254 990	255 087
12 345	12 350	12 400	12 000
963	960	1 000	1 060

3 The population of London in 1984 was 6696 000 (to the nearest thousand) round this to the nearest million.

Assume pulse rate is equal to heart rate.

4 Suppose a friend's pulse rate is 68 beats per minute. Use sensible rounding and answer these questions.

- (a) How many heart beats per day?
- (b) How many heart beats per week, month, year?
- (c) How many beats per lifetime?

Interpret your results as if to your friend.

2.1.2 Rounding decimals

Using a calculator often gives a long string of digits. For example, $1 \div 3$ might give .33333333. But very often, for practical purposes, this level of accuracy is too precise to be useful. You try measuring .33333333 metres!

Decimal numbers can be rounded just like whole numbers. For example, 0.33333333 is nearer to 0.3 than to 0.4 and rounding 0.33333333 to 0.3 m makes it easy to measure. Here the number has been rounded so that it contains one decimal digit; it has been *rounded to one decimal place*.

Often it is more sensible to measure to the nearest centimetre (0.01 m), in which case 0.333 3333 m rounds to 0.33 m. Here the number contains two decimal digits; it has been *rounded to two decimal places*.

The rule for rounding applies just as well to decimal places as hundreds.

The number of decimal places given, however, indicates the accuracy. So 0.30 indicates accuracy to the nearest 0.01 (two decimal places) whereas 0.3 indicates accuracy to the nearest 0.1 (one decimal place).

We often write 'd.p.' for 'decimal place'.

The general rule for rounding to a number of decimal places is to look at the digit one place to the right of the number of digits you want to round to, and round up or down depending on whether this digit is '5 or more' or '4 or less'.

Example 2

- (a) In a diary, the following conversion statements were given. Round all the numbers to one decimal place (1 d.p.) in order to give a handy measure to remember when travelling abroad:

converting miles to kilometres multiply by 1.6093;

converting feet to metres multiply by 0.3048.

- (b) Round these values so that they are correct to two decimal places (2 d.p.).

Solution

- (a) The answer must contain one decimal digit, so look at the second decimal digit and round up or down accordingly.

- miles to kilometres: the second decimal digit is 0 so round down, thus the conversion factor is 1.6 (to 1 d.p.).
- feet to metres: the second decimal digit is 0 so round down, thus the conversion factor is 0.3 (to 1 d.p.).

- (b) To round to 2 decimal places look at the third decimal digit:

- miles to kilometres: the third decimal digit is 9, so round up. Thus the conversion factor is 1.61 (2 d.p.).
- feet to metres: the third decimal digit is 4 so round down. Thus to convert feet to metres multiply by 0.30 (2 d.p.).

Note the use of a zero in the second decimal place. This is necessary to indicate that the value has been rounded to *two* decimal places.

Try some yourself (2.1.2)

Solutions on page 100.

- 1 Complete the following table.

Number	Rounded to		
	1 d.p.	2 d.p.	3 d.p.
0.47265			
13.959943			
56.09827			

- 2 The table below contains some errors. Identify and correct them.

Number	Rounded to		
	1 d.p.	2 d.p.	3 d.p.
3.1415926	3.10	3.14	3.150
22/7	3.1	3.04	3.142
0.019999	0.0	0.19	0.200

- 3 Imagine you are alone on an island and that your only source of drinking water is a wrecked ship's full water tank. It measures 4 metres long by 3.45 metres wide and is 2.84 metres high. Use reasonable rounded values to make an estimate (without using your calculator) of how long you can survive until it rains, assuming that you need 2 litres per day and that the water remains perfectly fresh. You may need the conversion 1 cubic metre contains 1000 litres.

Volume =
length $\times$ width $\times$ height.

2.1.3 Significant figures

Conversion of feet to metres was given in Example 2.

We often write 's.f.' for 'significant figures'.

Sometimes it doesn't make sense to round to a specific number of decimal places. If, say, you were calculating the cost of fencing at £10.65 per metre, for a garden boundary, the length of which had been given to you as 185 feet, then you would want to multiply $10.65 \times 185 \times 0.3048$. To do the calculation quickly, you might round this to $10 \times 200 \times 0.3 = 600$ and thus estimate the cost as about £600. (You might use this figure as a check on the exact calculation, done on a calculator.)

What has been done is round each of 10.65, 185, 0.3048 to *one significant figure*. Thus 10.65 is rounded to 10 (the 1 is the significant figure); 185 has been rounded to 200 (the 2 is the significant figure); and 0.3048 has been rounded to 0.3 (the 3 is the significant figure). All of these numbers contained zeros, but they were not deemed significant so far as the accuracy of the estimation was concerned. In the £600 answer, only the 6 is significant. You would expect the answer to be about six hundreds but not zero tens and zero units.

The first non zero digit (from the left) in a number is the first significant figure. To round to a given number of significant figures, first count from the first significant digit to the number required (including zeros). If the next digit is 5, 6, 7, 8 or 9 round up, otherwise round down.

Example 3

If you have your course calculator you may have found how to check the free space available in the memory. Suppose the display shows 2354 856 bytes. Express this available memory space correct to:

- (a) One significant figure (1 s.f.), which might be appropriate if chatting to a fellow student on the phone.
- (b) Two significant figures (2 s.f.), which might be appropriate if working out if there is enough memory for several large data sets.

Solution

- (a) One significant figure: the first digit (from the left) is the 2. The next significant digit is the 3, and since this is less than 5 round down. Thus the available memory space in the calculator is about 2000 000 bytes (1 s.f.).
- (b) Two significant figures: the third significant digit is the 5, so round up. Thus the available memory space in the calculator is about 2400 000 bytes (to 2 s.f.).

You can use the same procedure for numbers less than one.

Example 4

In scientific work people deal with very small units of measurement. Suppose you read that the spacing between adjacent atoms in a solid was 0.000 002 456 84 metres. You could make the number more memorable by using two significant figure accuracy.

Solution

To find 0.000 002 456 84 metres correct to two significant figures, count from the first non-zero digit on the left. The first significant digit in this number is the 2. The zeros to the left of this digit are not significant but they maintain the place value of the other digits. The second significant digit is the 4 and to the right of this digit is a 5 which means round up to give 0.000 0025 metres (2 s.f.).

You may have noticed that the word 'significant' has a more precise meaning in mathematics than when used in everyday language and that sometimes zeros are not counted as significant figures, even though they are obviously important in specifying the place value of a digit.

At the beginning of Subsection 2.1.3 there was a £600 estimate for the cost of fencing. The zeros are necessary to indicate that the 6 represents six hundreds, rather than six tens or six units, but the fact that they are zeros (rather than another digit) is not significant (in the exact cost, it is unlikely that both would be zero). However if an exact calculation gives a cost of £609, then the zero is significant. (To tell you it is six hundred, zero tens and nine units.) This is a price quoted to three significant figures.

For decimal numbers like 0.060 09 metres, the zeros to the left of the 6 are not considered significant (although they are important to indicate the place value), but those to the right are significant.

Try some yourself (2.1.3)

Solutions on page 100.

- 1 Complete the following table.

Number	Rounded to			
	1 s.f.	2 s.f.	3 s.f.	4 s.f.
2098 765				
0.042 6395				
0.000 289 99				

- 2 Complete the following table.

Number	Rounded to			
	nearest whole number	1 d.p.	1 s.f.	3 s.f.
238.483				
0.763 48				
0.088 8456				

- 3 On holiday it is often useful to do rough conversions into one's own currency to get a feel for how expensive an item is.

Suppose an Austrian Schilling (ASh) is £0.048654. Convert this to (a) one significant figure, (b) two significant figures, and in each case estimate the cost, in pounds, of an item costing ASh 278.

- 4 Has it ever occurred to you how far planet Earth has travelled through the solar system during the last year, relative to the Sun, and how fast we earthlings are travelling?

In order to estimate this it is necessary to make some assumptions:

- assume that the path of the Earth's orbit around the Sun is circular;
- the distance between the Earth and the Sun is approximately 92 860 000 miles, which is the radius of the orbit;
- distance travelled in one year is one orbit of the Sun;
- the distance travelled along a circular path is determined by working out the circumference of a circle. The relationship between circumference and radius is: $\text{circumference} = 2\pi \times \text{radius}$.

So

$$\text{distance travelled around Sun} = 2\pi \times 92\,860\,000 \text{ miles.}$$

Using a calculator π button gives

$$\text{distance} = 583\,456\,587.6 \text{ miles.}$$

- (a) For convenience round this to two significant figures and write this in words.
- (b) Given that the average speed is found by dividing the distance travelled by the time taken, estimate roughly how fast the earth is travelling along its orbit around the sun in miles per hour.

1 year = 8760 hours.

2.2 Estimation

Approximations are most useful when it comes to making rough estimates—like adding up a bill quickly to see if it is about right or checking a calculation. When using a calculator it is very important that you have an independent means (by estimation) of judging whether your answer is correct, or at least plausible. To make a rough estimate the numbers must be easy to work with, so being able to round numbers is an extremely useful skill.

Try these first

- 1 Measurement of a ceiling gives a length of 6.28 m and a width of 3.91 m.
- (a) Make a rough estimate of the area of the ceiling (the length times the width).
- (b) If one litre of paint covers roughly 11 square metres, roughly how much paint will you need to cover the ceiling?
- 2 Supposing a calculator was used to determine the area of the ceiling and the amount of paint needed in Question 1, and gave the answers:
- (a) 24.5548 square metres;
- (b) 270.1028 litres.

Do the answers agree with your rough estimates? Do the answers seem reasonable? If not, can you see what has gone wrong?

These are diagnostic questions. Try them to see which topics you need to revise.

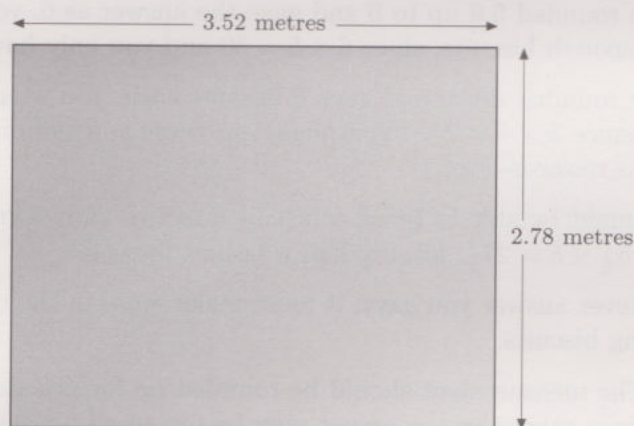
- 3 Suppose you went shopping with £30 cash and discount coupons to the value of £3.60 for various items that you intended to buy (such as £1 off a giant tin of biscuits). You did a rough estimate of your shopping bill by adding up the items you were buying (without taking the coupons into account) and it came to £25. Roughly how much cash would you expect to have left after paying the bill?
- 4 Suppose you are one of four people buying some joint presents for a family of five. The presents cost £2.15, £3.02, £2.99 and two at £2.50. Estimate your share of this cost.

However, when you used your calculator to calculate your share:

$$\frac{2.15 + 3.02 + 2.99 + 2 \times 2.50}{4},$$

using the calculator you got the answer £9.41. Why doesn't this answer make sense? Investigate what you might have done wrong.

- 5 Suppose you were sharing 28 biscuits among 5 children and you did the calculation $28 \div 5 = 5.6$. How many biscuits would you give each child?
- 6 Suppose you were considering purchasing a patterned carpet, priced at £15.75 per square metre, for the room shown in the diagram below which a friend has measured for you. It might be appropriate to use rounded values.
- (a) What degree of accuracy do you think would be most appropriate?



- (b) Estimate the cost of the carpet.
- (c) Would your estimate alter if the required carpet is from a roll 4 metres wide?
- 7 Find a rough estimate for each of the following calculations and then evaluate them on your calculator.
- (a) $0.91 + 2.8956$ (b) $0.6841 + 0.3692 + 0.2381$
- (c) 40.89×5.28 (d) $58.98 \times 82.93 \div 4.89$
- (e) $(4839 - 876) \times 0.0891$



Check your answers

- Follow-up in Section 2.2 **1** (a) Rounding to the nearest metre gives $6\text{ m} \times 4\text{ m}$, so the area is about 24 square metres.
- (b) Divide by the area covered by each litre (11 square metres). Rounding this to 10 square metres means that about $\frac{24}{10} = 2.4$ litres are needed. So two and a half litres is a good estimate.

- Follow-up in Section 2.3 **2** The first answer agrees with the rough estimate of 24 square metres. But the 270.1028 litres does not. Try to imagine 270 litres. It is an enormous quantity of paint. What has happened is that the area was multiplied by 11, instead of divided by 11.

- Follow-up in Section 2.3.1 **3** The amount of cash in pounds you will have left is roughly $30 + 3.60 - 25$, or equivalently $30 - 25 + 3.60$, i.e. about £8.60, or £9 (more roughly).

- Follow-up in Sections 2.3.2 and 2.3.3 **4** The cost of the presents is roughly $£2 + £3 + £3 + £5 (= 2 \times £2.50)$, which is £13. So your share is roughly $\frac{13}{4} = £3.25$. Another way of estimating is to notice that all the five presents cost two or three pounds, so the total cost will not exceed about $5 \times £3 = £15$. Therefore each of the four people's share of the cost must be less than £4 (since $4 \times £4 = £16$). In any event, £9.41 is clearly wrong.

What you have done is to calculate $2.15 + 3.02 + 2.99 + 2 \times 2.50 \div 4$, which your calculator interprets as $2.15 + 3.02 + 2.99 + (2 \times 2.50 \div 4)$. You need to work out $2.15 + 3.02 + 2.99 + 2 \times 2.50$ first and then divide this total by 4. This can be achieved using brackets: $(2.15 + 3.02 + 2.99 + 2 \times 2.50) \div 4$. (The exact answer is £3.29.)

- Follow-up in Section 2.3.3 **5** If you rounded 5.6 up to 6 and gave the answer as 6, you would not have enough biscuits, since $6 \times 5 = 30$ and you only have 28 biscuits.

If you rounded down and gave 5 biscuits each, you would have three over, since $5 \times 5 = 25$. (You might eat these yourself or keep them to give as rewards later.)

You might be able to break a biscuit into two and so answer $5\frac{1}{2}$, which gives $5\frac{1}{2} \times 5 = 27\frac{1}{2}$, leaving half a biscuit for you.

Whatever answer you gave, it must make sense in the context of sharing biscuits.

- 6** (a) The measurement should be rounded up for practical purposes, since otherwise the carpet may be too small for the room. An accuracy to the nearest 0.1 metre would be sufficient. Then the length of the room would be rounded to 3.6 metres and the width to 2.8 metres. (If the room is not exactly rectangular you might add on more for safety.)
- (b) To estimate the area and hence the price, you might round even more to 3.5×3 . These values give an area of 10.5 square metres. So round to 10. Round the price to £16 per square metre. So the estimate is £160.
- (c) If the carpet has a roll-width of 4 metres you have two options, depending upon whether the direction of the pattern on the carpet is important to you. This means you may need (approximately) either a 3 metre length (12 sq metres) or a length of 3.5 metres (14.5 sq metres). So you will need another 2 or 4 sq metres. This adds on about £30 or £60 giving an estimate of £190 or £220 respectively.

- 7 (a) Estimate: $1 + 3 = 4$. Calculation 3.8056.
 (b) Estimate: $0.7 + 0.4 + 0.2 = 1.3$. Calculation 1.2914.
 (c) Estimate: $40 \times 5 = 200$. Calculation 215.8992.
 (d) Estimate: $60 \times 80 \div 5 = 960$. Calculation 1000.24773.
 (e) Estimate: $4000 \times 0.1 = 400$. Calculation 353.1033.

For different purposes different degrees of approximation may be appropriate. Be careful about the level of accuracy of an estimate.

Example 5

Dave and Sally are planning to travel from Newcastle to London, then to Gloucester, then to Liverpool and finally home to Newcastle.

Newcastle to London 282

London to Gloucester 104

Gloucester to Liverpool 146

Liverpool to Newcastle 170

- (a) Make a rough estimate of the total distance travelled.
 (b) Petrol is estimated at £1 a litre. Their car does about 8 miles to the litre. Estimate their total petrol costs.

Solution

- (a) Get a rough estimate by rounding to the nearest 10.

Newcastle – London is about 280 miles.

London – Gloucester is about 100 miles.

Gloucester – Liverpool is about 150 miles.

Liverpool – Newcastle is about 170 miles.

So the total distance is about 700 miles. This can be written as

total distance $\simeq$ 700 miles.

(The same estimate, 700 miles, is also attained by rounding each figure to the nearest 100 in the first place! There are a number of ways of making rough estimates.)

- (b) To estimate the petrol costs, take the fuel consumption to be 10 miles per litre (as it is easier to divide by 10 than by 8). This gives the number of litres needed as about $700/10 = 70$ litres. At £1 per litre the cost is about £70.

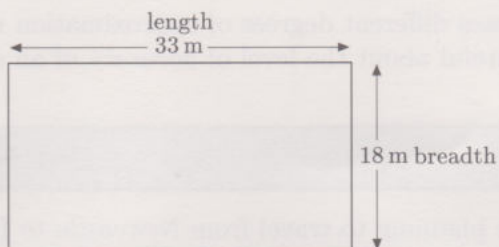
Since some information is given only to one or two significant figures, the estimate cannot be more accurate.

First make the numbers easy, then work out the estimate.

The sign $\simeq$ means 'is approximately equal to'.

Example 6

A rectangular lawn measures 33 m by 18 m. Make a rough estimate of its area. The lawn fertilizer to be used on it is suitable (when diluted) for about 100 square metres for each packet of fertilizer. How many packets are needed?



Solution

You get a rough estimate by rounding to the nearest 10:

33 rounds down to 30;

18 rounds up to 20.

So

$$\text{area} \simeq 30 \text{ m} \times 20 \text{ m}$$

$$\simeq 600 \text{ square metres.}$$

One packet of fertilizer covers about 100 square metres. So six packets will probably be needed.

These examples illustrate that a rough estimate can give a reasonable idea of the result.

An important use of estimation in MU120 is as a check for calculator work.

Example 7

Estimate $563.3 \times (6.891 + 44.34)$ as a check on the result you get by using your calculator to do the calculation.

Solution

This is a calculation you would be unlikely to undertake by hand. But, before you reach for your calculator, you should estimate your answer.

Round to the nearest whole number:

$$6.891 + 44.34 \simeq 7 + 44 \simeq 50$$

$$563.3 \simeq 600.$$

So the estimate of the calculation is

$$600 \times 50 = 30\,000.$$

So a rough estimate for the answer is 30 000.

You can use your calculator to find the answer exactly (using brackets). It comes to 28 858.4223, close to your estimate of 30 000.

If you got a very different answer, you would be suspicious and should look for an error.

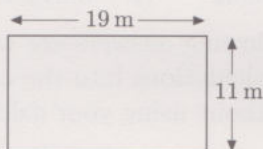


Evaluate the brackets first.

Try some yourself (2.2)

Solutions on page 101.

- 1 (a) Estimate a rough total for the supermarket bill below.
- | | |
|-------------|-----|
| margarine | 69p |
| milk | 22p |
| sugar | 49p |
| sausages | 92p |
| eggs | 87p |
| bread rolls | 97p |
- (b) Dave earns £784 per month. Make a rough estimate of how much he earns per year.
- 2 (a) Estimate the area of a strip of earth measuring 27 m by 3 m, which is to be grassed, in order to order the turf.
- (b) Make a rough estimate of the area of carpet tiles needed for the rectangular hall illustrated below.



- 3 During the summer season various music festivals are held. Glastonbury rock music and Cambridge folk are two of the larger events. Such large-scale events have planning implications for the organizers in respect of catering facilities, waste disposal and toilet/shower facilities.

Suppose that you are an organizer of the camping arrangements at such an event and you have sold 56 432 tickets in advance. So you know that a large crowd of people will be on location at your festival for at least three days. You will need to estimate how much field space you require and how many toilet/shower units you should hire.

This is a planning problem and sensible estimates are necessary in order to provide suitable facilities. A list of the assumptions that you may need to consider whilst solving this problem are given below.

- Most tent accommodation is shared by two people.
- A typical tent measures 3 metres by 2.5 metres and 2.2 metres high.
- Each tent requires a boundary of about 1 metre spacing surrounding it, to allow for tent pegs etc.
- Roughly one-fifth of the people will use the shower facilities in the peak demand time, between 7am and 10am.
- Each shower takes 5 minutes.
- On average people make about four visits per day to the toilet, spread evenly throughout day and night.
- Each toilet visit lasts roughly 2 minutes.

Make the following estimates:

- (a) Round the number of advanced ticket sales to one significant figure. Hence estimate the number of tents.
- (b) What area does each tent require in the field? So what is the area of field required?
- (c) Estimate the number of minutes of showering which are likely between 7am and 10am. How many minutes are there between 7am and 10am? About how many shower cubicles are needed?
- (d) How many toilet visits are there per 24 hours? How long is spent in total? So estimate how many toilet cubicles are required.
- (e) Consider the catering requirements. How many meals and cans of drink might be needed?



- 4 (a) For each of the following, make a rough estimate first; then find the answer using your calculator.
- (i) $75\,450 - 27\,654$ (ii) $676\,295 + 984\,122$
 - (iii) 16.859×4.962 (iv) $3.512 \div 720.5$
- (b) Some of the following answers are wrong, as the result of mistakes in keying the calculations into the calculator. Find which are the wrong ones, without using your calculator.
- (i) $631 - 529 = 6102$ (ii) $13\,726 + 4317 = 18\,043$
 - (iii) $1066 \times 45 = 7470$
 - (iv) $1066 \div 0.45 = 23.688\,889$ (rounded to six decimal places)



- 5 Find a rough estimate for each of the following and then evaluate on your calculator. In each case round your answer to two significant figures.
- (a) $0.086 + 219.2$ (b) $0.050 - 0.048$
 - (c) $457.3 - 64.5 + 302.4$ (d) 441.7×5.2
 - (e) $53.4 \times 70.9 \div 22.2$ (f) $217.5 + 60.3 \times 17.7$
 - (g) $(1285 - 329) \times 0.023$

2.3 Checking your answers

Once you have done a calculation, with or without the aid of a calculator, it is important that you pause for a moment to check your calculation.

You need to ask yourself some questions.

1. Have I done the right calculation in the right order?
2. Have I given due consideration to units of measurement?
3. Is my answer reasonable?
4. Did I make a rough estimate to act as a check?

Try this first

- 1 Use your estimating skills and common sense to check the following calculation, line by line, for the cost of papering an irregularly shaped room with simple non-patterned wallpaper.

$$\text{height of walls} = 275 \text{ cm}$$

$$\text{width of walls} = 512 + 346 + 234 + 748 \times 2 = 2588 \text{ cm}$$

$$\text{width of wallpaper} = 50 \text{ cm}$$

$$\text{number of pieces of wallpaper needed} = 2588 \div 50 = 51.76$$

$$\text{total length of wallpaper needed} = 51.76 \times 275 = 14\,234 \text{ cm}$$

$$\text{length of one roll} = 10 \text{ m}$$

$$\text{cost of one roll} = \text{£}4.00$$

$$\text{number of rolls} = 14\,234 \div 10$$

$$\text{total cost} = \text{£}4.00 \times \frac{14\,234}{10} = \text{£}5693.60$$

This is a diagnostic question. Try it to see which topics you need to revise.

Check your answer

- 1 The calculator calculations themselves are correct, but the total cost of over £5000 should suggest that something has gone wrong. The problem is that the length of a wallpaper roll is in metres (m) but the height of the room is in centimetres (cm). Hence the answer is out by a factor of 100. The correct answer for the total cost is therefore $\text{£}5693.6 \div 100 = \text{£}56.94$ (to the nearest penny).

Follow-up in Sections 2.3.3, 2.3.4, 2.3.5

Also, the number of pieces probably needs to be a whole number, as does the number of rolls. So these answers need to be rounded up:

$$\text{number of pieces} = 52$$

$$\text{number of rolls} = \frac{52 \times 275}{10 \times 100} = 14.3, \text{ rounded up to 15 rolls.}$$

Therefore the total cost is $\text{£}4.00 \times 15 = \text{£}60$ (rather than nearly £5000).

2.3.1 Have I done the right calculation?

Your calculation will probably involve a mixture of operations, like addition, subtraction, multiplication and division. You should check that you have used the right operations in the correct order.

Example 8

Suppose you are one of five people going out for a meal and sharing the total bill. Your friend Val, has done the accounts and shows you her calculation to check. Is she correct?

	£
Float from last outing	+0.40
Discount voucher for restaurant	+5.00
Contribution from Ali (who left early)	+8.00
Food	-36.98
Drinks beforehand	-5.10
Taxi	-3.50
Tip	-3.00
Total	-35.18
Share among the remaining four of us	÷4
	-8.795
Round up to nearest 10 p	-8.80
Everybody owes £8.80.	

Solution

Yes, she is correct.

Note that Val has used positive numbers to represent money in hand and negative numbers to represent debts or payments due, but she could have done the reverse. Sharing among four people, she has used division by four. It is important that the total was calculated before the division by four.

Example 9

- (a) Suppose somebody in the party queries the fact that Ali paid 80 p less than everybody else, saying that everybody should pay the same. Somebody else suggests the order of the calculation needs to be changed as shown. Is this correct?

	£
Float from the last outing	+0.40
Discount voucher for restaurant	+5.00
Food	-36.98
Drinks beforehand	-5.10
Taxi	-3.50
Tip	-3.00
Total	-43.18
Share among the remaining four of us	÷4
	-10.795
Round up to nearest £0.10	-10.80
Everybody owes £10.80	
Contribution from Ali (left early)	+8.00
Ali still owes £2.80	-2.80

- (b) Someone else, Paul, says, 'No, this is not right! Rather than redo the calculation, it would be simpler for Ali to share the 80 p which he underpaid among the rest of us, so he owes us 20 p each.' Is this a better solution?

Solution

- (a) There is something wrong here! The number of people sharing is now five not four and so you need to divide by five instead of four.

	£
Float from last outing	+0.40
Discount voucher for restaurant	+5.00
Food	-36.98
Drinks beforehand	-5.10
Taxi	-3.50
Tip	-3.00
Total	-43.18
Share among all five of us	÷5
	-8.636
Round up to the nearest 10 p	-8.70
Everybody owes £8.70	
Contribution from Ali (left early)	+8.00
Ali still owes 70 p	-0.70

- (b) Paul's solution is unfair to Ali. The result of that redistribution would be that everyone would have paid $£8.80 - 20\text{ p} = £8.60$, except for Ali who would have paid $£8 + 4 \times 20\text{ p} = £8.80$.

Try some yourself (2.3.1)

Solutions on page 102.

- In a supermarket the bill comes to £8.70, and you have discount coupons worth £3.50. The assistant says 'that will be £12.20 please'. Is she right?
- Suppose a young friend, Tom, has run some errands for you, for which you have given him £20 plus some discount coupons for the supermarket. He presents you with your shopping, the following calculation and £1 change.

received from you	£20.00
stamps from post office	£5.00 leaves £15.00
supermarket bill	£12.50 leaves £2.50
discount coupons	£1.50 leaves £1.00

Has Tom done the right calculation?

2.3.2 Have I used the correct order for my calculation?

When calculating an answer it is important that you give careful consideration to the order of operations used in the calculation. If you are using a mixture of operations remember that certain operations take priority in a calculation. Consider the following, apparently, simple sum.

$$1 + 2 \times 3 = ?$$

What answer would you give?

Did you give 7 as your response, or 9?

The correct answer is 7 but can you explain why?

If you have a calculator handy, check that it follows the correct order. Some cheap calculators do not, but the course calculator does.

As explained in Section 1.3, brackets come first, then powers, and then multiplication and division (working from left to right), and finally addition and subtraction (working from left to right).

Consider this sum:

$$10 - 5 + 3 = ?$$

What answer would you give?

Did you provide 2 as your response, or 8?

The correct answer is 8 but can you explain why?

The operations of addition and subtraction belong to the same group, and so you perform the calculation from left to right. Addition does not have priority over subtraction.

Consider these calculations:

$$(1 + 2) \times 3 = ?$$

$$10 - (5 + 3) = ?$$

These calculations are similar to earlier ones but this time involve the use of brackets, which take priority over the other operations. The correct answers here are 9 and 2 respectively.

Solutions on page 102.

Try some yourself (2.3.2)

1 Without using your calculator solve the following calculations.

- (a) $3 + 5 \times 2 = ?$ (b) $12 - 6 + 6 = ?$ (c) $6 + (5 + 4) \times 3 = ?$
(d) $(3 + 5) \div 2 = ?$ (e) $12 - (6 + 6) = ?$
(f) $6 + 5 + 4 \times 3 \div 2 = ?$

Check your answers using the course calculator.



2.3.3 Have I given due consideration to units of measurement?

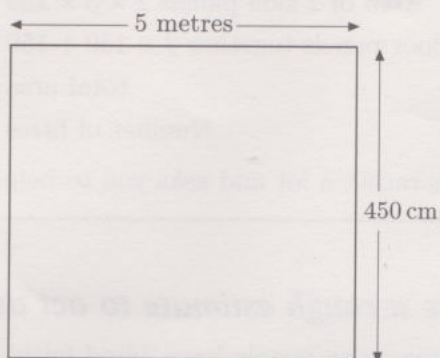
Many mathematical problems include units of measurement, as explained in Section 1.2. The measurement may be of length, weight, time, temperature or currency. The UK uses both metric and imperial units.

Make a list of the units of length with which you are familiar.

The table below gives the units that are in everyday use in the UK, but you may have included others.

<i>Metric</i>	<i>Imperial</i>
millimetre	inches
centimetre	feet
metre	yards
kilometre	miles

Think for a moment about other measurements; for example, those used for time, weight or capacity. There is a great variety of measurement units in use. Their incorrect use leads to errors which can sometimes be costly.

Example 10

Suppose that you want to buy a carpet, priced at £16.00 per square metre, for the room shown above and you calculate the area as follows.

$$\text{Area} = 5 \times 450 = 2250$$

$$\text{Cost of carpet} = 16 \times 2250 = \text{£}3600$$

An expensive carpet! What is wrong?

Solution

There is a mixture of units: metres and centimetres.

If a problem has a mixture of measurement units, it is usually a good idea to convert all measures to the same units. Here the carpet is sold in units of 'square metres' therefore ensure all the measurements are in metres.

$$\text{Hence area required} = 5 \times 4.50.$$

$$\text{Area} = 22.5 \text{ square metres.}$$

$$\text{Cost of carpet} = \text{£}16 \times 22.5 = \text{£}360.$$

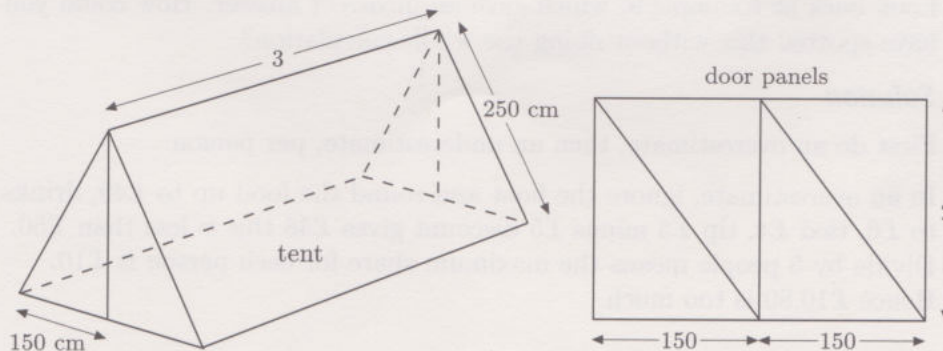
Try some yourself (2.3.3)

Solutions on page 102.

- 1 A friend has been quoted a price of £25.50 per square yard for tarmac surfacing of his yard. The yard measures 6 yards by 10 feet. Here is his calculation of the total cost. What is wrong with it?

$$\text{cost} = \text{£}25.50 \times 6 \times 10 = \text{£}1530$$

- 2 Another friend, Kim, wishes to reproof her tent. A litre tin of 'proofing agent' states that the coverage is roughly 5 square metres per litre. The manufacturer's measurements of the tent, which is a ridge tent, are: height 2 m, length 3 m. However, Kim realises that she needs the other measurements. Here is her diagram and calculation.



The problem: how many litres of proofing agent to buy to cover the tent?

$$\text{Area of 2 side panels } 2 \times 3 \times 250 = 1500$$

$$\text{Area of end door panels together } 2 \times 150 + 150 = 450$$

$$\text{total area} = 1950$$

$$\text{Number of litres} = 1950 \div 5 = 390$$

Kim thinks this is rather a lot and asks you to help. Find her errors.

2.3.4 Did I make a rough estimate to act as a check?

When using a calculator many people have 'blind faith' in its capacity to provide the correct result.

Calculators invariably provide the correct result for the information they are given; any errors are due to the operator.

To help guard against errors, always give a 'ball-park' estimate for a problem, using rounded values and easy calculations.

For example, make a rough estimate of the calculation below to act as a check upon the actual calculation using the calculator.

$$3.421 + 5.986 + 12.00987 = ?$$

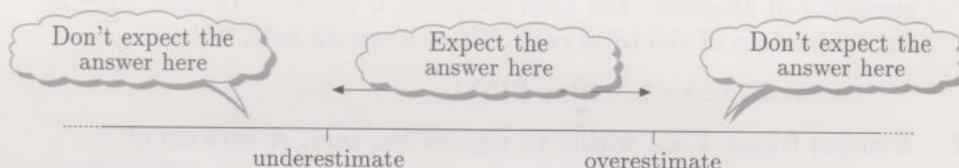
$$\text{Rough check: } 3 + 6 + 12 = 21$$

$$\text{Calculator answer: } 21.41687$$

The two answers are within the same 'ball-park' so you should be happy with the calculator's answer. Had you got an answer of 3438.99587, say, you should be suspicious and checking would reveal you had missed out a decimal point.

It is helpful to adopt this checking strategy for all your calculations since it is so easy to press the wrong key inadvertently, or double press a key when working at speed on a calculator.

Often it is useful to have both an underestimate and an overestimate.



Example 11

Look back at Example 9, which gave an incorrect answer. How could you have spotted this without doing the whole calculation?

Solution

First do an overestimate, then an underestimate, per person.

In an overestimate, ignore the float and round the food up to £40, drinks to £6, taxi £4, tip £3 minus £5 discount gives £48 this is less than £50. Divide by 5 people means the maximum share for each person is £10. Hence £10.80 is too much.

An underestimate would be £35 for food, £5 for drinks, £3 for taxi and £3 for tip, less £5 for the voucher and £1 float. This gives £40, divided by 5, which means minimum of £8 each.

Hence you would expect an answer between £8 and £10.

One common error when using a calculator is to forget that the calculator does not total everything as it goes along. So, for example, in the case of the bill for a night out, you must total everything before you divide by the number of people who are sharing the bill.

Example 12

Look back at Example 8 and suppose that the calculation was put into a calculator, in the following form:

$$0.40 + 5.00 + 8.00 - 36.98 - 5.10 - 3.50 - 3.00 \div 4 = ?$$

Would this give the right answer?

Solution

No, it would not give the right answer. The calculator will do the division first, i.e. $3.00 \div 4$, and then all the additions and subtractions, to give the answer -32.93. And £32.93 is clearly too much for each person's share of the bill.

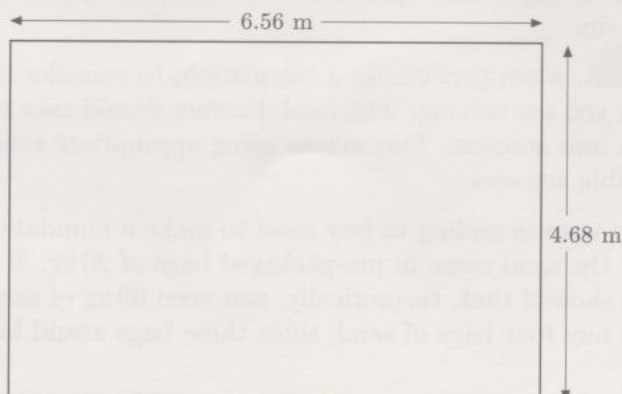
In order to make it do the calculation correctly, you can either total after the 3.00, by using the '=' button, or use brackets.

$$(0.40 + 5.00 + 8.00 - 36.98 - 5.10 - 3.50 - 3.00) \div 4$$

Try some yourself (2.3.4)

Solutions on page 103.

- 1 For each of the following calculations make suitable rough estimates before doing the calculation on your calculator and check the result.
 - (a) $22.12 \div 4.12$ (b) 0.897×10.00345
 - (c) $1009 \times 23.789 \times 278.98$ (d) $(12.98 + 14.87 + 63.098) \div 12.54$
- 2 You are helping a friend, Al, who is preparing a base for a new garage. He needs to purchase ready-made concrete from a local supplier. The concrete is sold in units of cubic metres so he needs to calculate how much to order.



The plan of the proposed garage is shown in the diagram above. Following advice from a builder the base is to have a thickness of 200 mm. Al's estimate for concrete required is given below.

$$\text{volume} = \text{length} \times \text{width} \times \text{thickness}$$

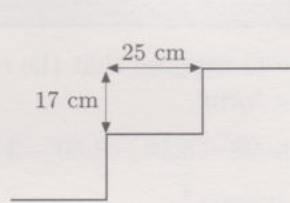
$$\text{volume} = 6.56 \times 4.68 \times 2.00$$

$$\text{volume} = 61.4016 \text{ cubic metres}$$

Hence volume of concrete needed is 61 cubic metres, to 2 s.f.

Make a rough estimate of what you would expect the answer to be. Are you happy with Al's calculation?

3



Suppose a friend has worked out the price of carpeting some stairs with carpet which costs £4.50 per square metre. The stairs are in three flights of ten, and each step requires 25 cm for the tread and 17 cm for the rise. The stair width is nearly one metre. There are also landings to be carpeted of length 1.2 m and 5.4 m, both one metre wide. By estimating what the answer should be, and then working through what he has done, check the calculation below for your friend.

$$\text{each stair } 25 + 17 = 42$$

$$\text{ten stairs } 42 \times 10 = 420$$

$$\text{three flights of stairs } 420 \times 3 = 1260$$

$$\text{landings } 1.2 + 5.4 = 6.6$$

$$\text{total length } 6.6 + 1260 = 1266.6$$

$$\text{cost } £4.50 \times 1266.6 = £5699.7$$

2.3.5 Does the answer make sense?

Having completed a calculation it is important to check whether your answer makes sense. British money for example can only be issued to the nearest penny. If you share a bill for £20 among three people, your calculator would give an answer of £6.66666667. So how much should be paid by each person?

You would probably round up to £6.67 or £6.70 and put the extra pence in a charity tin.

It is important, when performing a calculation, to consider the context of the problem you are solving. The final answers should take the context of the problem into account. This means using appropriate rounding in order to give sensible answers.

Suppose you were intending to buy sand to make a foundation for a path of slabs and the sand came in pre-packaged bags of 20 kg. If your calculations showed that, theoretically, you need 69 kg of sand, you would still have to buy four bags of sand, since three bags would be insufficient.

You might be able to break a biscuit into halves, but it is difficult to give six children 4.833 3333 biscuits each. Similarly, an answer which suggests you can afford to take 10.134 people on an outing is not very sensible.

When you get a mathematical answer, you need to interpret the answer within the context of the original problem and round your answer appropriately for the situation. So if you could afford to take 10.134 people on the trip, you would probably interpret this as ten people. As for 4.833 3333 biscuits, you would need to round down rather than up otherwise there would not be enough biscuits: four or four and a half might be all right, depending upon whether you could break them in half easily. (You could eat the remainder yourself.)

Example 13

Interpret the answers to the following sensibly.

- (a) Five people share a total bill for a night out of £56.66. They perform the calculation $56.66 \div 5 = 11.332$.
- (b) A children's charity has raised £1546.60 to pay for deprived children to go on holiday. Twelve children are selected to go. The calculation $1546.60 \div 12 = 128.883\ 3333$ is performed to see how much is available for each child.

Solution

- (a) To make sure that the bill is covered, the 11.332 needs to be rounded up so that each person pays £11.34. (The extra 4p collected could be added to the tip.)
- (b) This needs to be rounded down in order not to exceed the budget (unless further funds are available), and so something like £125 each (plus a contingency fund of £46.60) might be suitable.

The decision to make when giving a rounded answer to a problem is difficult to quantify and is often dependent upon previous knowledge or experience. Despite this difficulty it is always necessary to be conscious of the context of the problem you are solving and to give appropriate answers.

Try some yourself (2.3.5)

Solutions on page 103.

- 1 Complete the following table by rounding the given values appropriately:

Context	Appropriate rounding
Carpet floor area = 26.456 sq metres	
Interest earned = £109.876 5439	
Bill for £84.90 shared by 7 people	
Length of room = 5.387 6956 m	
Estimated attendance = 24.678	
Thickness of paper = 0.0543 mm	

- 2 A coach costs £85 to hire for a journey. There are six of you to share the cost. How much should you each pay? $85 \div 6 = 14.166\ 6667$

$$40.86 \times 3 \div 24 = 5.1075.$$

- 3 Suppose somebody baked two dozen fruit cakes for a charity sale and you want to charge a reasonable price for them. The cost of the ingredients, £40.86, is to be refunded to the cook. The organizer has suggested selling at a profit of about double this. What is a sensible selling price?
- 4 Suppose that a friend has asked you to check the following calculations he has made for material for new curtains. Is anything wrong?

$$\begin{array}{ll} \text{window 1: 2 curtains length 670 mm} & = 1340 \text{ mm} \\ \text{window 2: 4 curtains length 1 m 76 mm} & = 7.04 \text{ m} \\ \text{window 3: 3 curtains length 536 mm} & = \underline{1608 \text{ mm}} \\ & \text{total length} = 3652 \text{ mm} = 3.652 \text{ m} \end{array}$$

Cost at £3 per metre (length) is £10.956, i.e. approximately £11.

$$200 \div 35 = 5.714285714.$$

- 5 Suppose you are in charge of dealing the cards in a game for large numbers of people. You have a pack of 200 cards and 35 players. The instructions say: 'Deal as many cards as possible so that the players have the same number of cards each.' Somebody says you should give each player five cards. Is this correct?

Outcomes

Now that you have studied Module 2 you should be able to:

- ◇ round a given whole number to the nearest 10, 100, 1000 and so on;
- ◇ round a decimal number to a given number of decimal places or significant figures;
- ◇ use rounded numbers to find rough estimates for calculations;
- ◇ use the course calculator for decimal calculations involving +, −, × and ÷, giving your answer to a specified accuracy (e.g. decimal places or significant figures) and checking your answer by finding a rough estimate;
- ◇ check your answers to calculations by:
 - checking that the correct calculation has been done, in the correct order;
 - checking that the units are consistent;
 - checking the result against a rough estimate;
 - checking that the answer makes sense in the context of the problem.

Module 3 Ratio, proportion and percentages

The topics in this module, ratios, proportion and percentages, are concerned with dividing something into parts. For example, if there are 200 people living in a small village, and 50 of these are children, this could be expressed as a percentage:

25% of the village population are children;

or as a ratio:

one in every four people is a child *or* there is 1 child for every three adults;

or a proportion:

the proportion of children in the village population is a quarter.

One difficulty that some people find with these topics is knowing when to multiply and when to divide. This module revises the relevant techniques and helps you to apply them logically.

3.1 Ratio

Ratios crop up often in official statistics. The government wants the teacher–pupil ratio in schools to be increased to one teacher to thirty pupils or less. The birth rate has fallen: the ratio of children to women of child bearing age has gone down. It used to be 2.4 to 1, and now it is 1.9 to 1. Predictions for the ratio of working adults to retired adults is disturbing. Predictions are, that by 2030 the ratio will be two working adults to every retired person, instead of three to one now, and four to one ten years ago.

Often ratios are implicit in the language rather than explicitly referred to: one teacher for 30 pupils; 2.4 children per woman of child bearing age; one retired person per two working adults. The word ‘per’ often indicates that the concept of ratio is being used.

Try these first

- 1 A friend is painting the inside walls of a garage. So far she has used a 2 litre tin of emulsion paint and covered an area of 9 m^2 . She needs some more paint. How much more would you advise her to purchase if she intends to paint all the walls and ceiling, which is a total area of 75 m^2 .
- 2 A recipe for a casserole involves soaking dried beans. The beans require $1\frac{3}{4}$ litres of water per kilogram of beans. If you have $2\frac{1}{2}$ kilograms of beans, how much water is required?
- 3 A van driver averages 50 km per hour travelling on ordinary roads and 70 km per hour on motorways. Estimate:
 - (i) how far the van will travel in $2\frac{1}{2}$ hours;
 - (ii) how long it will take to travel 160 km;when travelling on (a) ordinary roads and (b) motorways.

These are diagnostic questions. Try them to see which topics you need to revise.

Check your answers

- Follow-up in Section 3.1.1 **1** An area of 9 m^2 requires 2 litres of paint. The ratio of paint to area is 2 to 9 or $\frac{2}{9}$. So an area of 1 m^2 would require $\frac{2}{9}$ of a litre of paint. She needs to paint an area of $(75 - 9) \text{ m}^2$ or 66 m^2 . This will require $66 \times \frac{2}{9} = \frac{132}{9}$ litres of paint (i.e. 14.666 67) litres or approximately 15 litres to the nearest $\frac{1}{2}$ litre (rounded up). So she needs 15 more litres of paint.
- Follow-up in Section 1.3.4 **2** You need to multiply $2\frac{1}{2}$ by $1\frac{3}{4}$. First convert to top-heavy fractions or decimals:

$$2\frac{1}{2} \times 1\frac{3}{4} = \frac{5}{2} \times \frac{7}{4} = \frac{35}{8} = 4\frac{3}{8}.$$
- Follow-up in Section 3.1.2 Alternatively $2.5 \times 1.75 = 4.375$.
 So $4\frac{3}{8}$ or 4.375 litres of water is required.
- Follow-up in Section 3.1.3 **3** (a) (i) In 1 hour the van travels 50 km so in $2\frac{1}{2}$ hours the van travels $50 \times 2\frac{1}{2} = 125 \text{ km}$.
 (ii) The van travels 50 km in 1 hour so it travels 1 km in $\frac{1}{50}$ hour. It therefore travels 160 km in $160 \times \frac{1}{50}$ hours, i.e. $\frac{16}{5}$ hours or $3\frac{1}{5}$ hours (3 hours 12 minutes).
 (b) (i) $70 \times 2\frac{1}{2} = 175 \text{ km}$.
 (ii) $\frac{160}{70} \approx 2.29$ hours or nearly 2 hours 20 minutes.

3.1.1 Expressing ratios

To make short crust pastry, one recipe book says 'use one part of fat to two parts of flour'; another recipe says 'use fat and flour in the ratio of one to two'; and yet another says 'use half as much fat as flour'. These are different ways of expressing the same ratio. Ratios are often expressed as fractions. So in this case:

$$\frac{\text{amount of fat}}{\text{amount of flour}} = \frac{1}{2}.$$

Equivalent fractions are discussed in Module 1.

Since you can multiply top and bottom of a fraction by the same number and get an equivalent fraction, you can use the ratio in a number of ways. If you have 100 grams of fat then

$$\frac{\text{amount of fat}}{\text{amount of flour}} = \frac{1 \times 100}{2 \times 100} = \frac{100}{200}.$$

So you need 200 grams of flour to 100 grams of fat. There are many ways to arrive at this answer. The important point is that a ratio of 100 to 200 is equivalent to 1 to 2.

To make concrete, the instructions are 'use sand and cement in the ratio three to one'. This means

$$\frac{\text{amount of sand}}{\text{amount of cement}} = \frac{3}{1}.$$

If you have 30 kg of cement, then you need 90 kg of sand.

$$\frac{3}{1} = \frac{3 \times 30}{1 \times 30} = \frac{90}{30}$$

The conversion rates between currencies or different units are often easier to remember as ratios. Many people remember that the ratio of distance in miles to the same distance in kilometres is five to eight.

Example 1

At the time of writing the ratio of prices in pounds sterling to prices in euros is two to three (2 : 3). What is the equivalent price in pounds for a coat costing 150 euros?

Solution

3 euros are equivalent to 2 pounds. This means that

$$\frac{\text{price in pounds}}{\text{price in euros}} = \frac{2}{3}.$$

$$\begin{aligned}\text{So price in pounds} &= \frac{2}{3} \times \text{price in euros} \\ &= \frac{2}{3} \times 150 = 100.\end{aligned}$$

The equivalent price of the coat is £100.

Time conversions are also ratios. The ratio of time measured in minutes to time measured in seconds is one to sixty (1:60), as there are sixty seconds in a minute.

Example 2

Adam's grandfather ran a mile in $4\frac{1}{2}$ minutes. Adam took 260 seconds. Which is greater, 260 seconds or $4\frac{1}{2}$ minutes? Did Adam run faster than his grandfather?

Solution

$4\frac{1}{2}$ minutes is equivalent to $4\frac{1}{2} \times 60$ seconds.

$4\frac{1}{2}$ is equivalent to $\frac{9}{2}$. So this is $\frac{9}{2} \times 60 = 270$ seconds.

So $4\frac{1}{2}$ minutes is just a bit longer than 260 seconds. Therefore Adam has run faster than his grandfather.

When shopping for a bargain, the ratio of price to quantity is often a useful way of comparing prices of different sized packets.

Example 3

A local shop sells ready-made custard at £1.45 for a special offer pack of three 425 g tins. It also sells the same brand of custard in 1 kg cartons costing £1.29 each. Which is the better bargain?

Solution

To compare the prices it would be best to compare the ratio of prices to amounts (measuring amounts in the same units) i.e. prices per kg.

Three 425 g tins will contain $3 \times 425 \text{ g} = 1275 \text{ g}$ or 1.275 kg.

£1.45 for 1.275 kg is $\frac{1.45}{1.275} \approx \text{£}1.14$ per kg.

The 1 kg carton costs £1.29 per kg.

The three tins are 15p cheaper per kg. So the three-tin pack is the better bargain.

Solutions on page 104.

Try some yourself (3.1.1)

- 1 If the ratio of distance measured in miles to the same distance measured in kilometres is five to eight, which is the higher speed limit, 70 miles per hour or 110 kilometres per hour?
- 2 A local supermarket sells a popular breakfast cereal in a 'Large Pack' and 'New Extra Large Pack'. They are both being sold at 'knock down' prices. The large pack contains 450 g of cereal priced at £1.85. The new extra large pack contains 625 g and is priced at £2.35. Which is the better bargain?
- 3 Baking potatoes are priced at 75 p for a pack of three (very similar) potatoes or at £2.70 for a 5 kg bag. How heavy does each of the three potatoes in the pack have to be in order for the pack to be a better bargain than the 5 kg bag? Does your answer seem reasonable? (Try to imagine a potato of this size.)

3.1.2 Converting ratios from fractions to decimals

Although ratios are often given as fractions, they can also be expressed as decimals. You need to deal with a mixture of fractions and decimals, and to compare ratios given in either form, so you need to be able to convert between the two forms.

Example 4

The ratio of the circumference of a circle to its diameter is a constant called pi (written π) which has been approximated by a number of different fractions. One such fraction is $\frac{22}{7}$, another is $\frac{355}{113}$. How do these compare with the decimal value from a calculator of 3.141 592 654?

Solution

If you have a calculator handy then you could key in $22 \div 7$ to convert to a decimal. However, if not, you might use long division or an informal method of division. Either way you should get 3.1428.... So $\frac{22}{7}$ agrees to 3 significant figures.

You will probably find it easier to use a calculator for dividing 355 by 113. The result is 3.14159292, which agrees to 7 significant figures.

Sometimes you are given a ratio as a decimal, but might find it easier to use and/or remember as a fraction.

Example 5

Suppose you had been told that the ratio between a distance measured in miles and the same distance measured in kilometres is about 0.625. Convert this to a fraction.

π is the Greek letter p .

Solution

$$0.625 = \frac{625}{1000}$$

This fraction is an adequate answer, or you can divide top and bottom by common factors to reach a simpler equivalent fraction.

Divide top and bottom by 5 to get $\frac{125}{200}$.

Divide top and bottom by 5 again to get $\frac{25}{40}$.

Divide top and bottom by 5 again to get $\frac{5}{8}$.

So the fraction is $\frac{5}{8}$, which is the fraction used earlier.

You might do this in one line cancelling

$$\frac{125 \overset{25}{\cancel{5}}}{200 \underset{40}{\cancel{5}}}$$

Try some yourself (3.1.2)

Solutions on page 104.

- 1 Convert each of the following fraction ratios to decimal ratios.
(a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $1\frac{1}{2}$ (d) $2\frac{1}{4}$
- 2 Convert each of the following decimal ratios to fraction ratios.
(a) 0.25 (b) 1.135 (c) 0.064

3.1.3 Speeds

Speed is the ratio of distance travelled to time taken. A runner's speed may be quoted in metres per second, miles per hour or kilometres per hour. The units are given as:

unit of distance *per* unit of time.

When you have a distance covered (such as a mile) and a time taken (such as four minutes) the **average speed** is defined as

$$\frac{\text{distance travelled}}{\text{time taken}}.$$

The formula for average speed applies even over a journey made up of several stages.

Example 6

In 1999, Hicham El Guerrouj held the record for running a mile. He covered the distance in just over 3 minutes 43 seconds. In the same year he also held the world record for the 1500 m race. He completed this distance in 3 minutes 26 seconds.

Work out:

- (a) his (average) speed in miles per hour for the 1 mile race;
- (b) his (average) speed in kilometres per hour for the 1500 metre race;
- (c) compare (a) and (b). In which race was his (average) speed faster assuming that 1.61 km = 1 mile?

Solution

- (a) To find the speed in miles per hour, you need the ratio of the distance in miles to time in hours. In 223 seconds (3 minutes 43 seconds) he ran 1 mile. Therefore in 1 second he would run $\frac{1}{223}$ miles. In 3600 seconds (i.e. 1 hour), he would run $3600/223$ miles i.e. 16.14 miles (to 2 d.p.). So the athlete's speed is 16.14 miles per hour (to 2 d.p.).
- (b) Hicham ran 1500 metres in 3 minutes 26 seconds (i.e. 206 seconds), so he ran $\frac{1500}{206}$ metres in 1 second. In 3600 seconds (1 hour) he runs $3600 \times \frac{1500}{206}$ metres. That is, 26 213.59 metres per hour. To convert to kilometres: divide by 1000 which gives 26.21 km per hour (to 2 d.p.).
- (c) To compare speeds in different units, you need to convert one to the other, say miles per hour to kilometres per hour. Since 1 mile is 1.61 km, 16.14 miles is $16.14 \times 1.61 = 25.99$ km (to 2 d.p.). Hence 16.14 miles per hour is the same as 25.99 km per hour. This is a lower speed than 26.21 km per hour. So Hicham ran the 1500 metre race faster, which is not surprising as it is a shorter distance.

Solutions on page 105.

Try some yourself (3.1.3)

- 1 Which is greater, 1.2 minutes or 70 seconds?
- 2 (a) A cheetah is the fastest land animal over short distances. It can run 400 metres in 15 seconds. What would be its speed in metres per second and kilometres per hour?
 (b) The slowest moving land animal is the three-toed sloth from tropical America. It takes 16 hours to travel a mile. What is this speed in kilometres per hour and in metres per second? (5 miles is approximately 8 km.)
- 3 Which is longer, 11 minutes or 0.17 hours?

3.2 Proportion

Proportion is another way of expressing notions of part and whole. You might say that the proportion of village inhabitants who are children is a quarter, or that the proportion of fruit juice in the punch is two thirds, or that the proportion of sand in the concrete is three quarters.

Or, using the language of ratios, you might say that the proportion of children in the village is one to four; that the proportion of fruit juice in the punch is two to three; that the proportion of concrete which is sand is three to four.

All these examples involve the fractions $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{4}$. Problems involving proportions are best handled by manipulating fractions, generally, by division or multiplication. The task is to decide which fraction to manipulate, in what way, and at what stage! These questions begin to help you decide.

Try these first

- 1 A recipe for four people calls for $\frac{3}{4}$ teaspoon of mustard powder. How much should you use for ten people?
- 2 Two workers in the Open University warehouse take 20 minutes to stick labels on 500 packages for an MU120 mailing. There are still 4000 more packages. How many workers are required, if the job is to be done in about a further hour?

These are diagnostic questions. Try them to see which topics you need to revise.

Check your answers

- 1 For one person you need $\frac{3}{4} \div 4 = \frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$ teaspoons. For ten people you need $\frac{3}{16} \times 10 = \frac{30}{16} = \frac{15}{8} = 1\frac{7}{8}$ teaspoons. So almost 2 teaspoons of mustard are needed.

Follow-up in Section 3.2.1

- 2 One worker should take twice as long as two. So 1 worker takes 40 mins for 500 labels. 4000 is 8 times 500 so 1 worker takes 40×8 min = 320 mins. 1 hour is 60 mins.

Follow-up in Section 3.2.2

So you need $\frac{320}{60}$ workers = $5\frac{1}{3}$.

Practically, you could have 5 workers for $\frac{320}{5} = 64$ mins (just over an hour) or 6 workers for $\frac{320}{6} = 53$ mins (just under an hour).

3.2.1 Direct proportion

In a recipe the quantity of each ingredient needed depends upon the number of portions. As the number of portions increases, the quantity required increases. The quantity per portion is the same. This is called direct proportion. The quantity is said to be **directly proportional** to the number of portions. If 2 potatoes are required for one portion, 4 will be required for two portions etc. A useful method for direct proportion problems is to find the quantity for one and multiply by the number you want.

Example 7

John lives with three cats. His daughter asks him to look after her cat for a week while she goes away. John normally buys two tins of cat food a day for the three cats. How many tins should he buy for the four cats for a week?

There are several ways to do this. Here is one.

Solution

3 cats eat 2 tins a day

1 cat eats $\frac{2}{3}$ tin a day

4 cats eat $4 \times \frac{2}{3} = \frac{8}{3}$ tins a day

4 cats eat $7 \times \frac{8}{3} = \frac{56}{3} = 18\frac{2}{3}$ tins a week

So John should buy 19 tins for the week.

Simplify the problems by considering 1 cat, rather than going straight to 4 cats.

Example 8

Debbie is checking her phone bill. Her mobile phone calls have all been charged at the same rate, 30 pence per minute. (Call charges are rounded to the nearest penny and charged to the nearest second.)

- She wants to check the cost of a call to her friend. The call lasted 7 minutes and 34 seconds. How much should she have been charged?
- How long, at this rate, can she speak to her friend if the call charge is to cost no more than £2.50?

Solution

- 1 minute cost 30p. (i.e. 60 seconds for 30p).
1 second for $\frac{30}{60}$ p = 0.5p.
7 minutes 34 seconds is 454 seconds, costing $454 \times 0.5\text{p} = 227\text{p}$.
The call should have been charged at £2.27.
- 0.5p will allow her to talk for 1 second.
1p will allow her to talk for 2 seconds.
£2.50 (250p) will allow her to talk for 250×2 seconds, = 500 seconds, i.e. 8 minutes 20 seconds.

3.2.2 Inverse proportion

In Section 3.3.1 you saw that direct proportion described relationships between two quantities, where as one increased, so did the other. Sometimes as one quantity increases the other decreases instead of increasing. This is called inverse or indirect proportion. Team tasks are often an example of this. The time taken to do a job is indirectly proportional to the number of people in the team.

A difficulty with the real-life context of such problems is that, in many cases, it is hard to believe that people working in a team will work at the same rate regardless of the size of the team, unless the team work independently, i.e. 'in parallel'. The main idea behind this type of problem is that increasing the number of people working decreases the time taken to complete the task. Such problems can be compared with certain problems involving speed: doubling the number of people working is the same as doubling the speed at which the team work. In either case the time is halved. It is useful to find out how long it would take one person to do the whole job, then divide by the number of people sharing the work. This is a good approach to most indirect proportion problems.

Example 9

A team of five people can deliver leaflets to every house on a housing estate in three hours. How long will it take a team of just two people?

Solution

Take the same approach as in Examples 7 and 8, and first work out how long it would take *one* person to deliver leaflets to the estate.

It will take one person five times as long as a team of five people. (If you find this hard to accept, imagine that the estate consists of five streets, and that each person delivers leaflets to one of these streets in the three

An obvious exception to this is decision-making in a committee: if two people can reach a decision in an hour, four people are liable to take twice as long!

hours.) So each street takes 3 hours to leaflet. It would take one person 5×3 hours to leaflet all five.

So it takes one person 15 hours to deliver leaflets to the whole estate. Two people will take half this time, so two people take $7\frac{1}{2}$ hours.

As a check: you would expect two people to take longer than five.

Try some yourself (3.2)

Solutions on page 105.

- 1 The length of time it takes to cook a Christmas pudding in a pressure cooker depends on the weight of the pudding. Kim has forgotten the time per pound but remembers that her $1\frac{1}{2}$ kg pudding takes 5 hours.
How long will it take to cook a 2 kg pudding?
- 2 A piece of computer software is to be developed by a team of programmers. It is estimated that a team of four people would take a year. Which of the following times is the length of time taken by three programmers?
A 1 year 3 months B 9 months C 1 year 4 months
- 3 A 10 kg bag of potatoes lasts for a week when used in catering for 7 people.
 - (a) How long will it last for 2 people, assuming everybody eats the same amount?
 - (b) If, instead of buying a 10 kg bag (which might not keep well), you want to buy fresh potatoes every week, how much per week should you buy for 2 people?

3.3 Percentages

Percentages are used, particularly in newspaper articles, to indicate fractions (as in '64% of the population voted') or to indicate changes (as in 'an increase of 4%').

Try these first

- 1
 - (a) What is 40% as a fraction?
 - (b) What is $\frac{3}{4}$ as a percentage?
 - (c) What is 1.26 as a percentage?
- 2 In an election 1248 votes out of 3000 are cast for the Gold party. What is this as a percentage? If in the previous election, 35.1% voted Gold, what is the swing to Gold in percentage points?
- 3 A quote from a builder for some home improvements is £2200 plus VAT.
 - (a) If VAT is $17\frac{1}{2}\%$, what is the amount including VAT?
 - (b) If the builder gives 10% reductions for prompt payment, what would the amount be then (including VAT)?

These are diagnostic questions. Try them to see which topics you need to revise.



Check your answers

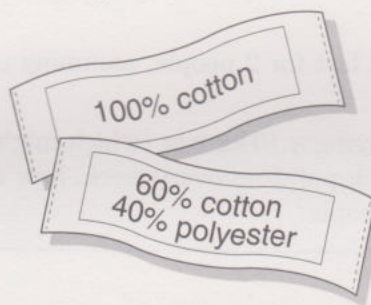
- Follow-up in Section 3.3.1 1 (a) $40\% = \frac{40}{100} = \frac{2}{5}$.
 (b) $\frac{3}{4} = \frac{3}{4} \times 100\% = 3 \times 25\% = 75\%$.
 (c) $1.26 = 1.26 \times 100\% = 126\%$.

- Follow-up in Section 3.3.1 2 $\frac{1248}{3000} \times 100\% = 41.6\%$.
 The swing is $41.6 - 35.1 = 6.5$ percentage points.

- Follow-up in Section 3.3.2 3 (a) $\text{£}2200 \times \frac{117.5}{100} = \text{£}2585$.
 (b) $\text{£}2585 \times \frac{90}{100} = \text{£}2326.50$.

3.3.1 What are percentages?

Percentages often indicate proportions. For example, labels in clothes indicate the various proportions of different yarns in the fabric. 'Per cent' means 'per hundred', and is denoted by the symbol %. 100% is the same as the whole, or one hundred per hundred.



100% cotton indicates that the fabric is made entirely from cotton. (100 parts out of 100 parts).

60% cotton means that $\frac{60}{100}$ (or 0.6) of the fabric is cotton.

40% polyester means that $\frac{40}{100}$ (or 0.4) is polyester.

The percentages on the label should total 100%, just as the corresponding fractions add up to 1, because the total (100%) refers to the whole garment. $60\% + 40\% = 100\%$, $0.6 + 0.4 = 1$.

Percentages can also be manipulated as either fractions or decimals.

Example 10

A building society offers 90% mortgages to first-time buyers. How much would the Smiths get on a house valued at £150 000?

Solution

They want to find 90% of £150 000. $90\% = \frac{90}{100} = 0.9$.

$$0.9 \times 150\,000 = 135\,000$$

So the Smiths would get £135 000.

Section 1.5 in the *Calculator Book* will show you how to handle percentages on the course calculator.

Fractions and decimals can also be converted to percentages, by multiplying by 100%.

So, for example, 0.17, 0.3 and $\frac{3}{4}$ can be expressed as percentages as follows:

$$0.17 \times 100\% = 17\%;$$

$$0.3 \times 100\% = 30\%;$$

$$\frac{3}{4} \times 100\% = 75\%.$$

Decimals or fractions bigger than 1 correspond to percentages greater than 100%. For example,

$$1\frac{3}{4} = 1.75, \text{ which as a percentage is } 1.75 \times 100\% = 175\%.$$

Care needs to be taken when talking in percentages. A percentage is a percentage of a given quantity: 50% of the voters, 25% of the budget, 10% of the population. In newspapers, it is not always clear what quoted percentages are percentages of. Politicians too can give misleading statements. 'We are giving an increase in funding of 10% and 5% of this has no strings attached'. In this statement it is not clear whether the 5% is 5% of the original funding or 5% of the 10% increase in funding (which would be $\frac{5}{100} \times 10\% = 0.5\%$ of the original funding).

To avoid this confusion some people use the term percentage points, when they are comparing the percentages of different quantities. In elections the percentage of the vote at one election for a given party is often compared with the percentage of the vote at the previous election. The difference is referred to as the 'swing' and expressed in percentage points.

Example 11

In a local election 2540 votes are cast for the Purple party out of a total of 5000 votes. At the previous election 42.1% of votes had been for the Purple party. What is the swing to the Purple party in percentage points?

Solution

2540 out of 5000 as a percentage is $\frac{2540}{5000} \times 100 = 50.8\%$

Swing = $50.8 - 42.1 = 8.7$ percentage points.

Try some yourself (3.3.1)

Solutions on page 106.

- 1 (a) Express each of the following percentages as fractions.
 (i) 40% (ii) 8% (iii) 70% (iv) $5\frac{1}{2}\%$
- (b) Express each of the following percentages as decimals.
 (i) 50% (ii) 85% (iii) 7% (iv) $17\frac{1}{2}\%$
- (c) A survey was carried out on 840 heterosexual couples to investigate family income. 75% of the women were in paid work; 90% of the men were in paid work; 18% of the couples had a joint income of less than £160 per week.
 - (i) How many women were in paid work?
 - (ii) How many men were in paid work?
 - (iii) How many couples had a joint income of less than £160 per week?

Round off the percentages to whole numbers.

- 2 Convert each of the following to percentages.
 (a) (i) 0.8 (ii) 0.21 (iii) 0.70 (iv) 2.4
 (b) (i) $\frac{1}{2}$ (ii) $\frac{1}{8}$ (iii) $\frac{1}{9}$ (iv) $1\frac{1}{7}$
- 3 In the election in Example 11 the Average party gets 2100 votes. In the previous election they had 10%. Find the swing.

3.3.2 Percentage increase and decrease

Our everyday experience of percentages includes percentage increases (like VAT at $17\frac{1}{2}\%$, or a service charge of 15%) and percentage decreases (such as a discount of 15%).

For example, £8 plus $17\frac{1}{2}\%$ VAT means you actually have to pay

$$£8 + (17\frac{1}{2}\% \text{ of } £8).$$

$17\frac{1}{2}\%$ of 8 is $\frac{17.5}{100} \times 8 = \frac{140}{100} = 1.4$ so the VAT is £1.40, and the sum you pay is $£8 + £1.40 = £9.40$.

There is another way of doing this calculation:

100% of £8 plus $17\frac{1}{2}\%$ of £8 = $(100\% + 17\frac{1}{2}\%)$ of £8,
 that is 117.5% of £8.

$$\begin{aligned} 117.5\% \text{ of } £8 &= £(1.175 \times 8) \\ &= £9.40 \text{ (as before).} \end{aligned}$$

Discount can be calculated in the same way. For example, £8 with 15% discount means you actually pay

$$\begin{aligned} &£8 \text{ less } (15\% \text{ of } £8) \\ 15\% \text{ of } 8 &= \frac{15}{100} \times 8 = \frac{120}{100} = 1.2. \text{ So the discount is } £1.20 \\ £8 - £1.20 &= £6.80 \end{aligned}$$

Alternatively the actual amount can be calculated in another way, as follows:

$$\begin{aligned} (100\% - 15\%) \text{ of } £8 &\text{ is } 85\% \text{ of } £8. \\ 85\% \text{ of } £8 &= £(0.85 \times 8) \\ &= £6.80 \text{ (as before).} \end{aligned}$$

Example 12

A restaurant bill comes to £76 before VAT and the service charge are added. VAT is added at $17\frac{1}{2}\%$ and the restaurant also adds a service charge of 15%. Does it make any difference to what you have to pay if the VAT is added first *then* the service charge or vice versa? Does it make a difference to the amount of VAT paid?

Solution

Adding VAT first gives

$$117\frac{1}{2}\% \text{ of } £76 = 1.175 \times £76 = £89.30.$$

Adding the service charge on to £89.30 gives

$$115\% \text{ of } £89.30 = 1.15 \times £89.30 = £102.70 \text{ (correct to the nearest penny).}$$

So the total bill is £102.70. Adding these extras the other way round:

$$115\% \text{ of } £76 = 1.15 \times £76 = £87.40;$$

$$117\frac{1}{2}\% \text{ of } £87.40 = 1.175 \times £87.40 \\ = £102.70 \text{ (correct to the nearest penny).}$$

The total bill is the same. The order does not matter from the customer's point of view. However, it does from the VAT collector's point of view (and indeed from that of the restaurant management, who receive a bigger service charge if it is calculated last). If VAT is added first, VAT is $17\frac{1}{2}\%$ of £76. If VAT is added last, VAT is $17\frac{1}{2}\%$ of £87.40. Hence legally VAT must be added last!

$$117\frac{1}{2}\% = 100\% + 17\frac{1}{2}\%$$

$$115\% = 100\% + 15\%$$

The first case

$1.15 \times 1.175 \times £76$ and in the second $1.175 \times 1.15 \times £76$.

The order doesn't matter in multiplication:

$$1.175 \times 1.15 = 1.15 \times 1.175.$$

Example 13

- A carpet store is offering 20% off all its oriental rugs. What would the sale price be for a Chinese rug originally priced at £245?
- The business fails to do well and decides to close down. It makes a further reduction on all its stock of 12%. What would the same Chinese rug be sold for now?

Solution

- In the first sale the Chinese rug would cost $(100\% - 20\%)$ of £245, $100\% - 20\% = 80\%$, $0.80 \times £245 = £196$.
- In the final closing sale the rug would cost $(100\% - 12\%)$ of £196 or $0.88 \times £196 = £172.48$.

Example 14

The number of casualties handled by the outpatients department of a hospital increases by approximately 8% per year. The number of casualties this year was 1920. Make a prediction for the number of casualties handled (a) next year, (b) in two years' time.

Solution

- Casualties would be $100\% + 8\%$ of 1920, that is, 108% of 1920.

This gives $1.08 \times 1920 = 2073.6 = 2074$ (correct to nearest whole number) as a prediction for the number of casualties next year.

- Repeat this calculation again on next year's figures to find the predicted figure for the number of casualties in two years' time.

This is 108% of 2074. $1.08 \times 2074 = 2239.92 \approx 2240$ casualties.

Solutions on page 106.

Try some yourself (3.3.2)

- 1 (a) How much will this tennis racquet cost if VAT at $17\frac{1}{2}\%$ has to be added?



- (b) The racquet is put in the sale at 10% discount. What is its sale price, not including VAT?
- (c) If customers pay VAT *and* get the discount, how much do they pay? Does it make any difference to the customer whether the VAT is added first then the discount subtracted, or vice versa? Give a reason for your answer.
- 2 A new train operator boasts 'Train times reduced by 12%'. Decrease 90 minutes by 12%. Give your answer as minutes and seconds.
- 3 A commuter pays £1260 for a season ticket. The train company announces an increase of 7.4% on all its fares. How much will the season ticket cost after the increase?
- 4 An airline decides to increase its fares from London to Europe by 17% and its fares to North America by 15%. Before the increase a special offer European fare from London to Geneva was £165 and the North American fare from London to New York was £376. What would the new fares be for these trips after the increases?
- 5 The population of a small town is 4650. It is predicted that the population will decrease by approximately 4% each year. What is the population likely to be after (a) 1 year, (b) two years?

Outcomes

Now that you have studied Module 3 you should be able to:

- ◇ work with simple ratios;
- ◇ convert between fractions, decimals and percentages;
- ◇ explain the meaning of ratio, proportion and percentage;
- ◇ find percentages of different quantities;
- ◇ calculate percentage increases and decreases;
- ◇ calculate average speeds in given units and find speeds, distances and times for travel at constant speed;
- ◇ convert units;
- ◇ solve problems involving direct and indirect proportion.

Module 4 Squares, roots and powers

This module reminds you about powers of numbers, such as squares and square roots. In particular, powers of 10 are used to express large and small numbers in a convenient form, known as *scientific notation*, which is used by the course calculator.

4.1 Squares, cubes and roots

This section reminds you of the meaning of squares and square roots, cubes and cube roots.

Try these first

- 1 Square carpet tiles measure 50 cm by 50 cm.
 - (a) What area is covered by one tile? Give your answer (i) in square centimetres (cm^2) and (ii) in square metres (m^2).
 - (b) What area, in m^2 , will 100 tiles cover?
 - (c) How many tiles are needed to cover 1 square metre?
 - (d) How many tiles are needed to cover a room with an area of 16 m^2 ?
 - (e) If the room in (d) is square, how many tiles go along each edge?
- 2 Find the volume of a one-foot cube in cubic metres (1 foot = 30.48 cm). Estimate your answer first. Round your answer to three decimal places.
- 3 Without using your calculator, estimate the following, as preparation for the calculator work in Question 4.
 - (a) 9^2 (b) 4^3 (c) $\sqrt{64}$ (d) $(\frac{1}{3})^3$ (e) $(-3)^2$
- 4 Use your calculator to find:
 - (a) 9.42^2 (b) 3.65^3 (c) $\sqrt{70}$ (d) 0.33^3 (e) $(-2.713)^2$Use your answers to the previous question as rough checks. Round your answers to four decimal places.

These are diagnostic questions. Try them to see which topics you need to revise.



Check your answers

- 1 (a) (i) One tile covers $50 \times 50 = 2500 \text{ cm}^2$.
(ii) To convert this to m^2 , you could divide by 100^2 to give
$$\frac{2500}{100^2} = \frac{2500}{10\,000} = 0.25 \text{ m}^2$$
. Alternatively (and more simply) convert first: each tile measures $0.5 \text{ m} \times 0.5 \text{ m}$ and so has an area $0.5 \text{ m} \times 0.5 \text{ m} = 0.25 \text{ m}^2$.
(b) One tile covers 0.25 m^2 so 100 tiles cover $100 \times 0.25 = 25 \text{ m}^2$.
(c) Since each tile covers 0.25 m^2 , 4 tiles will cover $4 \times 0.25 = 1 \text{ m}^2$.
(d) 4 tiles cover 1 m^2 so $16 \times 4 = 64$ tiles cover 16 m^2 .
(e) If the room is square and the area is 16 m^2 then each edge must be 4 m, since $4 \times 4 = 16$ (or $\sqrt{16} = 4$). So 8 tiles go along each edge. (You could also have deduced this from your answer to (d), since 64 tiles is 8 tiles $\times$ 8 tiles.)

Follow-up in Section 4.1.1.

Follow-up in Section 4.1.3. **2** To estimate an answer, choose a simple approximation, say

$$1 \text{ foot} \simeq 30 \text{ cm} = 0.3 \text{ m.}$$

$$\text{Then } 1 \text{ foot cubed} \simeq (0.3)^3 = 0.027 \text{ m}^3.$$

$$\text{More accurately, } 1 \text{ foot} = 30.48 \text{ cm} = 0.3048 \text{ m.}$$

$$\text{So } 1 \text{ foot cubed} = (0.3048)^3 = 0.028 \text{ to three d.p.}$$

Follow-up in Sections 4.1.1, 4.1.2, 4.1.3.

3 (a) $9^2 = 9 \times 9 = 81$ (b) $4^3 = 4 \times 4 \times 4 = 16 \times 4 = 64$

(c) $\sqrt{64} = 8$ (since $8^2 = 64$) (d) $(\frac{1}{3})^3 = \frac{1^3}{3^3} = \frac{1}{27}$

(e) $(-3)^2 = -3 \times -3 = 9$

Follow-up in Sections 4.1.1, 4.1.2, 4.1.3.

4 (a) $9.42^2 = 88.7364$ (b) $3.65^3 \simeq 48.6271$

(c) $\sqrt{70} \simeq 8.3666$

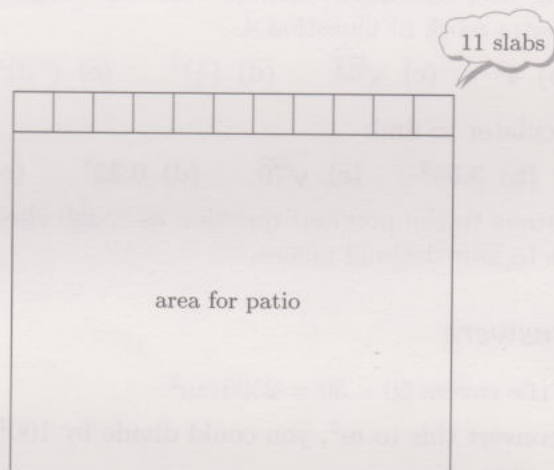
(d) $0.33^3 \simeq 0.0359$ (check this by calculating $\frac{1}{27}$, the estimate from 3(d) above, as a decimal.

(e) $(-2.713)^2 \simeq 7.3604$ (N.B. To get this answer on the course calculator, you need to key in the brackets.)

4.1.1 Squares

Example 1

A gardener is planning to create a square patio using concrete slabs. A row of 11 large slabs just fits across the width of the areas she wants to cover.



- How many rows of slabs will she use?
- How many slabs will she need altogether?
- The slabs are 0.75 m square. What area will the patio be?

Solution

- (a) Since the patio is square and it is 11 slabs wide, it must be 11 slabs long too, so there are 11 rows of slabs.
- (b) That means that altogether she will need 11×11 , or $11^2 = 121$ slabs.
- (c) There are several ways of finding the area of the patio. One method is to say that each slab is 0.75 m by 0.75 m, which is an area of 0.75^2 square metres. So the patio is $11^2 \times 0.75^2$ square metres. Using a calculator gives 68.0625 square metres. An alternative method is to say that each side of the patio is 11×0.75 m. so the area of the patio is $(11 \times 0.75)^2$ square metres. Using a calculator again gives 68.0625 square metres. So the patio will be just over 68 square metres.

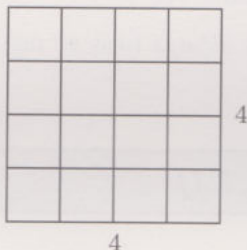
Notice

$$(11 \times 0.75)^2 = 11^2 \times 0.75^2.$$

In general, to **square** a number, multiply it by itself. This is denoted by writing a small '2' to the top right of the number,

e.g. 4 squared, written 4^2 , is $4 \times 4 = 16$.

There is a squaring key on your calculator—see Section 1.4 of the *Calculator Book*.

**Example 2**

In an art course the teacher asks each of the pupils to draw a sketch of every child in the class, including a self-portrait, on separate sheets of paper. There are 26 children in the class. Find an upper estimate for the number of sheets of drawing paper he should supply for the whole class. Find a lower estimate. Use your calculator to find the exact number of sheets of paper.

Solution

Each of the 26 children will need 26 pieces of paper, so altogether 26×26 , i.e. 26^2 sheets of paper are required.

Upper estimate: If there were 30 pupils in the class he would need $30 \times 30 = 900$ sheets of paper.

Lower estimate: If there were only 20 pupils he would need $20^2 = 400$ sheets of paper.

So he will need something in between 400 and 900.

On the calculator, $26^2 = 676$.

This is in reasonable agreement with the rough check, so 676 sheets of paper are required. (He would probably take 700 to have spares.)



You have now seen how to find squares of whole numbers and decimals. What about fractions? The rule is as before: to square a fraction, just multiply it by itself.

For example:

$$\left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

In Example 1, you could have used $\frac{3}{4}$ instead of 0.75. Check on your calculator that $0.75^2 = \frac{9}{16}$. Both answers should be 0.5625.

There may be contexts where you need to square negative numbers. Take care when doing so. Use the brackets if in doubt. Calculating by hand gives, for example:

$$(-9)^2 = -9 \times -9 = 81$$

However, keying in -9 and then squaring will give -81 on the course calculator. Why? It is calculating exactly what you have asked it for, i.e. -9^2 . Since 9^2 is 81, -9^2 is -81. The number 9 is squared *first*, to give 81, and *then* made negative, to give -81. Brackets are needed to tell the calculator to make the 9 negative and *then* square:

$$(-9)^2 = 81,$$

but without brackets -9^2 means $-(9^2)$.

To avoid any ambiguity, it is *always* best to use brackets when squaring negative numbers.

Section 1.4 of the *Calculator Book* shows you how to use the course calculator to square a negative number.

Some other calculators do give $-9^2 = 81$.

Solutions on page 107.

Try some yourself (4.1.1)

- 1 Without using your calculator, find the following.
 (a) 10^2 (b) 100^2 (c) 0.1^2 (d) 0.01^2
 (e) $\left(\frac{1}{2}\right)^2$ (f) $(-3)^2$ (g) $(-1)^2$
- 2 Explain why brackets should be used when writing the expression $(-1)^2$.
- 3 Use your calculator to find the following, using your answers to the appropriate parts of Question 1 as estimates.
 (a) 96^2 (b) 10.65^2



4.1.2 Square roots

Given any number, you now know how to find its square. But, given the squared number, how do you find the original number?

Example 3

If the gardener in Example 1 had only 49 paving slabs, what size of square patio could she make?

Solution

You probably spotted that 49 is 7×7 , or 7^2 , so she could make a square patio 7 slabs by 7 slabs.

Since $7^2 = 49$, 7 is the **square root** of 49, written

$$7 = \sqrt{49}.$$

Sometimes you can just look at a number and spot its square root, if the number is a 'perfect square' (i.e. the result of squaring a whole number). For example, 25 is a perfect square, and $\sqrt{25} = 5$. But more often than not you will need to use your calculator for square roots, and it is important to be able to find rough estimates as a check on your calculator work. So if you wanted $\sqrt{55}$, you would know that it would lie between $\sqrt{49} = 7$ and $\sqrt{64} = 8$, and you would expect an answer of seven point something. (It is 7.416 ...).

Technically, -7 is also a square root of 49, since $(-7)^2 = 49$. This is called the negative square root. The sign $\sqrt{}$ is customarily used to denote the positive square root, so $\sqrt{49} = 7$ and $-\sqrt{49} = -7$.

In Example 3, only the positive square root is relevant (patios have positive length sides).

There is a square root key on your calculator. Square roots are covered in Section 1.7 of Chapter 1 of the *Calculator Book*.

Example 4

The owners of a new house, with a bare earth garden, see an advertisement for 44 square metres of turf, 'free to a good home – only pay transportation'. They were planning a square lawn surrounded by flower beds.

Find a rough estimate for the square root of 44. Use your calculator to find $\sqrt{44}$ and find the size of the square lawn which the turf would make.

Solution

Rounding 44 down to 40 doesn't help—you don't know the square root of 40 either! But you do know that $6^2 = 36$ (which is less than 44) and $7^2 = 49$ (which is greater than 44) so $\sqrt{44}$ lies between 6 and 7. You could leave the answer as 'between 6 and 7' or guess it as 6.5, say. The calculated answer is 6.6332 (rounded to four decimal places). So the turf would make a lawn about 6.6 m square.

Try some yourself (4.1.2)

Solutions on page 107.

- 1 The new home owners from Example 4 above want to price grass seed, as well as the turf (transport only). The best buy seems to be loose seed, which says '1 kilo covers 80 m²'. They wonder what length the side of an 80 m² square lawn would be? Make an estimate and then use your calculator to find $\sqrt{80}$ and hence the side of the square lawn.
- 2 Use your calculator to find each of the following, estimating the answer first in each case.
 (a) $\sqrt{144}$ (b) $\sqrt{14.4}$ (c) $\sqrt{1.44}$ (d) $\sqrt{0.144}$

Where necessary, round your answers to four decimal places.

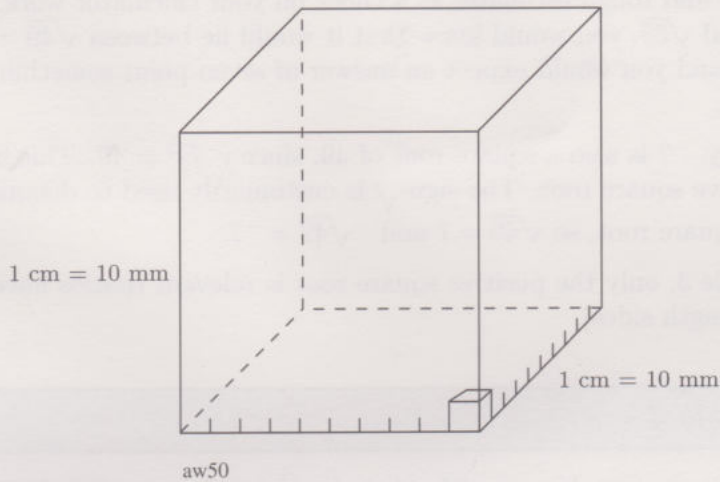


4.1.3 Cubes

In the same way that square units are used to measure area, cubic units are used to measure volume. A cube measuring $1\text{ cm} \times 1\text{ cm} \times 1\text{ cm}$ has a volume of 1 centimetre cubed, or 1 cubic centimetre, written as 1 cm^3 , or 1 cc. The volume of this cube in millimetres is

$$1\text{ cm}^3 = 10\text{ mm} \times 10\text{ mm} \times 10\text{ mm} = 1000\text{ mm}^3.$$

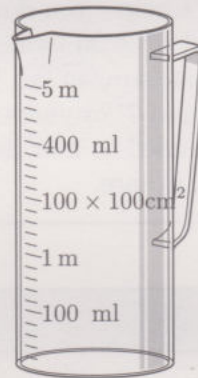
Volume of a 'box' is length $\times$ width $\times$ height.



Example 5

My half-litre measuring jug is marked off in divisions of 100 ml, with subdivisions of 20 ml. I want to measure out 420 cc, but the only conversion table I have tells me that a litre is one cubic decimetre. How many cubic centimetres are there in a litre? Can I use the measuring jug for 420 cc?

One decimetre is a tenth of a metre.



Measuring jug

Solution

There is one cubic decimetre in a litre, so 1 litre measures $1\text{ dm} \times 1\text{ dm} \times 1\text{ dm}$ (if shaped into a cube). There are 10 centimetres in a decimetre, so a 1 litre cube would measure

$$10\text{ cm} \times 10\text{ cm} \times 10\text{ cm} = 1000\text{ cm}^3.$$

So 1 litre is 1000 cm^3 . But there are 1000 millilitres in a litre, so it turns out that a millilitre is the same as a cubic centimetre.

$$1\text{ ml} = 1\text{ cm}^3.$$

So I can use the jug to measure 420 ml, or 420 cc (or cm^3).

$$1\text{ m} = 10\text{ dm} = 100\text{ cm}.$$

This is a useful result:
1 litre = 1000 cm^3 .

1 cubic centimetre can be written as 1 cc or 1 cm^3 .

To find the cube of a number, multiply three copies of it together. For example:

$$2^3 = 2 \times 2 \times 2 = 4 \times 2 = 8$$

$$(-2)^3 = -2 \times -2 \times -2 = 4 \times -2 = -8$$

You can use your calculator to find cubes. 2^3 is 'two cubed' or 'two to the power three'. Just as 'square root' is the opposite process to squaring, so 'cube root' is the opposite process to cubing. $4^3 = 64$, so $\sqrt[3]{64} = 4$.

See Section 1.4 of the *Calculator Book*.

Try some yourself (4.1.3)

Solutions on page 107.

- 1 If a litre is one cubic decimetre, how many litres are there in a cubic metre?
- 2 Find the following without using your calculator, as an estimate for the calculator work in the next question.
(a) $(-1)^3$ (b) 3^3 (c) 100^3 (d) 0.1^3
- 3 Use your calculator to find the following.
(a) $(-1.2)^3$ (b) 3.3^3 (c) 101^3 (d) 0.121^3
- 4 What is:
(a) $\sqrt[3]{1}$; (b) $\sqrt[3]{1000}$?

4.2 Powers

Try these first

These are diagnostic questions. Try them to see which topics you need to revise.

- 1 The size of a population of micro-organisms doubles every hour. If there are two of these creatures to start with, how many will there be after five hours?
- 2 Without using your calculator, find the following, as estimates for calculation work in the next question.
(a) $(-1)^5$ (b) 3^4 (c) 2^{-2}
- 3 Use your calculator to find the following.
(a) $(-1.3)^5$ (b) 3.2^4 (c) 2.2^{-2}
(Use your answers to the previous question as rough checks.) Round your answers to four decimal places.
- 4 Express as powers of ten
(a) a million (b) a thousand
(c) a million divided by a thousand



Check your answers

- 1 After one hour there are $2 \times 2 = 4$ micro-organisms.
After two hours there are $2 \times 4 = 8$, or $2 \times 2 \times 2 = 2^3$.
:
After five hours there are $2 \times 32 = 64$, or $2 \times 2 \times 2 \times 2 \times 2 = 2^6$.
So there will be $2^6 = 64$ after five hours.

Follow-up in Section 4.2.1

If you gave the answer 32, notice that there were *two*, not *one*, to start with.

- | | | |
|----------------------------|----------|--|
| Follow-up in Section 4.2 | 2 | (a) $(-1)^5 = -1$ (any odd power of -1 is -1)
(b) $3^4 = 3 \times 3 \times 3 \times 3 = 81$
(c) $2^{-2} = \frac{1}{2^2} = \frac{1}{4} (= 0.25)$ |
| Follow-up in Section 4.2.1 | 3 | (a) $(-1.3)^5 \simeq -3.7129$ (b) $3.2^4 = 104.8576$
(c) $2.2^{-2} \simeq 0.2066$ |
| Follow-up in Section 4.2.2 | 4 | (a) a million $= 1\,000\,000 = 10^6$ (b) a thousand $= 1\,000 = 10^3$
(c) 10^6 divided by $10^3 = 10^{6-3} = 10^3 =$ a thousand |

4.2.1 Power notation

Here is a tale based on an ancient Eastern legend, which gives an idea of the impact of raising a number to a power.

Example 6

A long time ago there lived a very rich king whose son's life was saved by a poor old beggar woman. The king was naturally very grateful to the woman, so he offered to give her anything that she wanted. Much to the king's surprise the old lady just asked for two bags of rice. When asked why she specifically wanted the rice, she explained that she would like to have her son to stay and that if she had some rice she would be able to feed him. The king felt that this was far too small a reward so he asked if there were any other members of her family that she might like to have to stay. She replied that she would like to have her two daughters the following week, so four bags of rice then would be a great help. The king agreed to this and suggested that she might like to think of having more of her family to stay. He said he would be prepared to double the number of bags he gave her each week. At this point his accountant got very agitated and asked him to consider very carefully what he was offering! What was he worried about?

Solution

In order to decide if the king was being exceptionally generous, you need to look at exactly what he was proposing. In the beginning the numbers are quite small, 2 bags of rice in the first week, 4 bags in the second, then 8 in the third, 16 in the fourth, 32 in the fifth and so on. However, to get a feel for how the numbers are growing, look at them in another way.

Week 1 2 bags of rice = 2
Week 2 2×2 bags of rice = 4
Week 3 $2 \times 2 \times 2$ bags of rice = 8
Week 4 $2 \times 2 \times 2 \times 2$ bags of rice = 16
Week 5 $2 \times 2 \times 2 \times 2 \times 2$ bags of rice = 32
⋮

It is not particularly convenient to write out all these 2s, particularly if you want to go on and look at the situation in week 52, say, i.e. a year later. However, there is a short and neater way of expressing these numbers.

You can write $2 \times 2 = 2^2 = 4$ and $2 \times 2 \times 2 = 2^3 = 8$.

In a similar manner, in week 4 you can write $2 \times 2 \times 2 \times 2 = 2^4 = 16, \dots$ and in week 10, when there are ten twos multiplied together, you can write 2^{10} . Sometimes you may find it written $2 \wedge 10$ ($\wedge$ is the 'power of' sign).

In week 1, when the number of bags is just 2, you can write this as 2^1 .

These numbers can be worked out on your calculator; you will find instructions in Section 1.4 of the *Calculator Book*. Check each of 2^4 , 2^5 and 2^{10} using your calculator.

You should find that $2^{10} = 1024$ and so see what the accountant was worried about.



The notation in Example 6 is called **power notation**, or **index notation**. In a number such as 2^5 , the 5 is called the **power**, or **index**, of the number.

The squares that were dealt with in Section 4.1 are particular examples of powers: 9^2 , for example, can be thought of as '9 to the power 2'.

For most numbers, calculating powers by hand soon becomes tedious: while you might be quite happy to find 2^5 or 9^2 , it would take a long and fairly dull time to find 2^{50} or 9^{20} by hand. So you will be using your calculator for most calculations involving powers, even when the numbers themselves are quite simple. However, there is one number whose powers are quite easy to find, namely 10. For example:

$$\text{one hundred} = 10 \times 10 = 10^2 = 100;$$

$$\text{one thousand} = 10 \times 10 \times 10 = 10^3 = 1000;$$

$$\text{one million} = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6 = 1000\,000.$$

It is also easy to find powers of 1 and 0.

$$1 \times 1 \times 1 \times \dots = 1.$$

$$\text{and } 0 \times 0 \times 0 \dots = 0.$$

In the same way that 2 can be written as 2^1 so 10 can be written as 10^1 .

Example 7

The headings on the place value tables in module 1 were tens, hundreds, thousands etc. Write these as powers of ten. What do you think the units columns heading would be as a power of ten?

Solution

ten	$= 10 = 10^1$
hundred	$= 100 = 10^2$
thousand	$= 1000 = 10^3$
ten thousand	$= 10\,000 = 10^4$
hundred thousands	$= 100\,000 = 10^5$
million	$= 1\,000\,000 = 10^6$

The power of ten increases for each column. So to be consistent, the units column should be 10^0 . Putting 10^0 into a calculator gives $10^0 = 1$. So the units column is $1 = 10^0$.

Solutions on page 107.

Try some yourself (4.2.1)



- 1 Find the following powers by hand, as estimates for calculator work.
(a) 10^7 (b) 10^8 (c) 3^4 (d) $(-2)^2$ (e) $(-2)^3$
- 2 Use your calculator to find the following and use your answers to the previous question as a check. Give your answers correct to 3 significant figures.
(a) 11^7 (b) 10.8^8 (c) 3.14^4 (d) $(-2.2)^2$ (e) $(-2.01)^3$
- 3 (a) How many zeros are there in 10^{12} when written out in full? Work out a rule for finding the number of zeros in any positive whole number power of 10.
(b) Your answers to parts (d) and (e) of Question 1 may have suggested a rule for powers of negative numbers, depending on whether the power is odd or even. What is the rule?
- 4 Hamsters of a particular kind, left to breed in a suitable environment, double their population every month. If a friend of yours, who is thinking of breeding these hamsters, started with one pair (male and female), what population might she expect in a year's time (assuming she keeps them all)?
- 5 A ploughed field has been left abandoned. It has one dandelion in the middle. Dandelion flowers have about 1000 seeds each. If a tenth of these germinate to produce one flower each in a year's time, estimate the possible number of dandelion flowers in the field, if it is abandoned for five years.
- 6 Cells in a laboratory culture divide into two every day, given sufficient nutrient. If there were five cells on day one, how many would there be on day ten?

4.2.2 Multiplying and dividing powers

Powers of ten can be used to investigate what happens when two powers of the same number are multiplied together. For example, consider multiplying 10 by 100:

$$\begin{aligned}
 10 \times 100 &= 10^1 \times 10^2 \\
 &= \underbrace{10}_{1 \text{ ten}} \times \underbrace{10 \times 10}_{2 \text{ tens}} \\
 &= \underbrace{10 \times 10 \times 10}_{3 \text{ tens}} \\
 &= 10^3 \\
 &= 1000.
 \end{aligned}$$

A billion is a thousand million. In terms of powers this is:

$$\begin{aligned}
 1000 \times 1000\,000 &= 10^3 \times 10^6 \\
 &= \underbrace{10 \times 10 \times 10}_{3 \text{ tens}} \times \underbrace{10 \times 10 \times 10 \times 10 \times 10 \times 10}_{6 \text{ tens}} \\
 &= \underbrace{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}_{9 \text{ tens}} \\
 &= 10^9 \\
 &= 1000\,000\,000
 \end{aligned}$$

so a billion is 10^9 or 1000 000 000.

An American billion is a thousand million. In UK a billion used to be a million million, but now the American billion is the standard usage.

The number of zeros in the whole number power of ten is the same as the power.

In the above calculations, $10^1 \times 10^2 = 10^3$ and $10^3 \times 10^6 = 10^9$, and the way these have been worked out shows that, when you *multiply* numbers expressed as powers of ten, you *add* the powers.

Powers of 10 were used because they are easy to work with, but you can just as easily show that multiplying powers of the same number always means adding the powers. For example:

$$7^2 \times 7^3 = \underbrace{7 \times 7}_{2 \text{ sevens}} \times \underbrace{7 \times 7 \times 7}_{3 \text{ sevens}} = \underbrace{7 \times 7 \times 7 \times 7 \times 7}_{5 \text{ sevens}} = 7^5$$

so $7^2 \times 7^3 = 7^{2+3} = 7^5$. Check this on your calculator!

To multiply two powers of the same number, add the powers.

Now consider division in the same way. For example;

$$\frac{1000}{100} = \frac{10^3}{10^2} = \frac{10 \times 10 \times 10}{10 \times 10} = 10 = 10^1$$

$$\frac{1000\,000}{100} = \frac{10^6}{10^2} = \frac{10 \times 10 \times 10 \times 10 \times 10 \times 10}{10 \times 10} = 10\,000 = 10^4.$$

So $10^3 \div 10^2 = 10^1$ and $10^6 \div 10^2 = 10^4$, which suggests that dividing the powers of ten means subtracting the powers. You can see that the two tens underneath have cancelled with two of the tens on top in each case, which shows why the number of tens underneath is subtracted from the number on top. This argument applies equally well to any number. For example:

$$7^5 \div 7^3 = \frac{\overbrace{7 \times 7 \times 7 \times 7 \times 7}^{5 \text{ sevens}}}{\underbrace{7 \times 7 \times 7}_{3 \text{ sevens}}} = 7 \times 7 = 7^2.$$

Work out $7^5 \div 7^3$ on your calculator to check this.

To divide two powers of the same number, subtract the powers.

Now, what happens if you divide a number by itself? You should get the answer 1. For example

$$\frac{1000}{1000} = 1, \quad \text{but} \quad \frac{1000}{1000} = \frac{10^3}{10^3} = 10^{3-3} = 10^0;$$

so 10^0 must be equal 1. This must be true for any numbers (other than 0). So, for example, $7^0 = 1$, $123^0 = 1$, $1000^0 = 1$. It even works for negative and decimal numbers. Check a few numbers on your calculator to see that raising to the power 0 gives 1.



0^0 is a problem!

Now look at what happens when the power is negative. What does 10^{-3} mean? What is the result of the following calculation?

$$100 \div 100\,000$$

What you are actually being asked to find is:

$$\frac{100}{100\,000} = \frac{1}{1000} = 0.001$$

But look at the calculation again. Using the rule for the division of powers of numbers gives:

$$10^2 \div 10^5 = 10^{2-5} = 10^{-3}$$

So $10^{-3} = 0.001$. But you can also write this result as:

$$\frac{1}{1000} = \frac{1}{10 \times 10 \times 10} = \frac{1}{10^3}$$

This means that 10^{-3} can be thought of as 1 divided by 10^3 . Similarly

$$10^{-1} = \frac{1}{10} = 0.1 \text{ and } 10^{-2} = \frac{1}{100} = 0.01.$$

This result is true for all negative powers, not just powers of ten. For example:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8} = 0.125$$

One over any number is called the **reciprocal** of the number. So, for example, the reciprocal of 10 is $\frac{1}{10} = 10^{-1}$ and the reciprocal of 100 is

$$\frac{1}{100} = \frac{1}{10^2} = 10^{-2}.$$

A number raised to a negative power is the reciprocal of the number raised to the corresponding positive power.

$$\text{So } 10^{-5} = \frac{1}{10^5} = 0.00001.$$

Example 8

In Module 1, the decimal place value table columns are headed units, tenths, hundredths, thousandths etc. Write these as powers of ten.

Solution

$$\text{a tenth} = \frac{1}{10} = 10^{-1}$$

$$\text{a hundredth} = \frac{1}{100} = 10^{-2}$$

$$\text{a thousandth} = \frac{1}{1000} = 10^{-3}$$

So the headings of the place value table are all powers of ten. To the left of the units they are positive powers, the units column is 10^0 , and to the right the column headings are negative powers of ten.

...	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	...

Notice that

$$10^{-1} = \frac{1}{10} = 0.1$$

$$10^{-2} = \frac{1}{100} = 0.01.$$

The negative power indicates the position of the decimal point — how many times it has moved to the left from $10^0 = 1$.

$$10^{-1} = 0.1 \text{ (move one to the left); } 10^{-2} = 0.01 \text{ (move 2 to the left) etc.}$$

Reciprocals are covered in Section 1.7 of the *Calculator Book*.

Try some yourself (4.2.2)

Solutions on page 108.

- 1 Find each of the following by hand, giving your answers both as a power of ten and as a decimal number. You will use these answers as a check on your calculator work in the next question.
 (a) 10^{-2} (b) $10^2 \times 10^3$ (c) $10^7 \div 10^4$ (d) $10^4 \div 10^7$
- 2 Evaluate the following, using your answers from Question 1 as estimates.
 (a) 10.3^{-2} (to four significant figures)
 (b) $10.1^2 \times 10.11^3$ (to four significant figures)
 (c) $10.112^7 \div 10.21^4$ (to one decimal place)
 (d) $10.12^4 \div 10.351^7$ (to three significant figures)
- 3 Find each of the following by hand.
 (a) 1024^0 (b) 1024^1 (c) 5^{-1} (d) 10^{-4}
- 4 How many zeros are there after the decimal point in the number 10^{-6} ?
 How many zeros are there after the decimal point for any given negative power of 10?
- 5 (a) Explain why a negative power of a number is one divided by the corresponding positive power of that number.
 (b) Use the power button on your calculator to find the following.
 (i) $\frac{1}{4}$ (ii) $\frac{1}{10^2}$

Hint: remember that a number to the power zero is 1.

4.3 Scientific notation**Try these first**

- 1 The diameter of the Sun is 1 392 000 km. Write this, in metres, in scientific notation.
- 2 The centre of the Milky Way galaxy is 2.6×10^4 light years away from Earth. One light year is 9.46×10^{12} km. How far away is the centre of the Milky Way in kilometres? Estimate your answer first and then use your calculator to do the calculation, giving your answer in scientific notation.
- 3 Evaluate $0.000\,923 \times 0.007\,67$. Estimate your answer first (using scientific notation), then use your calculator.

These are diagnostic questions. Try them to see which topics you need to revise.



A light year is the distance light travels through space, in one year.

**Check your answers**

- 1 1392 000 km is 1392 000 000 m (multiplying by 1000). In scientific notation this is
 1.392×10^9 m.
 (Alternatively: 1392 000 km is
 1.392×10^6 km = $1.392 \times 10^6 \times 10^3$ m = 1.392×10^9 m.)

Follow-up in Section 4.3.1

Follow-up in Section 4.3.2. **2** Estimate: The centre is roughly 3×10^4 light years away, and each light year is roughly 9×10^{12} km, so the distance to the centre is approximately $3 \times 10^4 \times 9 \times 10^{12} = 27 \times 10^{16} = 2.7 \times 10^{17}$.

Calculate:

$$2.6 \times 10^4 \times 9.46 \times 10^{12} = 2.4596 \times 10^{17}.$$

Follow-up in Section 4.3.2 **3**

$$0.000\,923 \times 0.007\,67 = 9.23 \times 10^{-4} \times 7.67 \times 10^{-3}$$

Estimate:

$$\begin{aligned} &\simeq 9 \times 10^{-4} \times 8 \times 10^{-3} = 72 \times 10^{-7} \\ &\simeq 7 \times 10^{-6}. \end{aligned}$$

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Calculate:

$$0.000\,923 \times 0.007\,67 = 7.079\,41 \times 10^{-6}.$$

4.3.1 Expressing numbers in scientific notation

In Module 1 you looked at place values for numbers, and in Section 4.2 you saw why they were called powers of ten.

Place value	10 000	1000	100	10	1	0.1	0.01	0.001	0.0001	0.000 01
Power of ten	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}

Using this notation, very large numbers or small decimal numbers can be expressed in a particularly neat form. This is how the course calculator displays numbers which will not fit on the screen. This is illustrated by the following examples:

$$\begin{aligned} 10^7 &= 10\,000\,000 & \text{so} & \quad 25\,000\,000 & = 2.5 \times 10^7 \\ 10^{-6} &= 0.000\,001 & \text{so} & \quad 0.000\,0025 & = 2.5 \times 10^{-6} \end{aligned}$$

A number expressed in this form is said to be in **scientific notation**. Any decimal number can be converted to scientific notation, which is

$$\boxed{\text{any number}} = \boxed{\begin{array}{c} \text{number} \\ \text{between} \\ 1 \text{ and } 10 \end{array}} \times \boxed{\begin{array}{c} \text{power} \\ \text{of } 10 \end{array}}$$



The course calculator automatically uses scientific notation when the answer to a calculation is very large or very small. For example, return to our tale of the king and the bags of rice. In week 52 the woman would receive 2^{52} bags. Calculate this on the course calculator. You should get 4.503 599 627 E 15. This answer means $4.503\,599\,627 \times 10^{15}$. To get a feel for how big this number is, i.e. for how many bags of rice the king would need to provide a year later, look at some other numbers in scientific notation:

Great Britain has an area of $229\,850\text{ km}^2$.

In scientific notation, $229\,850 = 2.2985 \times 100\,000 = 2.2985 \times 10^5$.

The distance from the Earth to the Sun is $149\,600\,000\text{ km}$.

In scientific notation, $149\,600\,000 = 1.496 \times 100\,000\,000 = 1.496 \times 10^8$.

Consider now the bags of rice calculation. 10^{15} is 1000 000 000 000 000. So the number of bags of rice is 4503 599 627 000 000. This is a very large

number of bags of rice! The king would need to give the woman over four thousand million million bags of rice in week 52. No wonder his accountant was concerned.

As mentioned above, scientific notation is also used for very small numbers. Try finding $\frac{5}{16\,000\,000}$ on your calculator. You should find that the answer on your calculator is $3.125 \text{ E } -7$. This is 3.125×10^{-7} . This time the index is negative. This is the same as:

$$3.125 \times \frac{1}{10^7} = \frac{3.125}{10\,000\,000} = 0.000\,000\,3125.$$

With practice you may not need the intermediate steps. You may be able to just move the decimal point the appropriate number of places (multiplying by 10^{-3} means moving the decimal point 3 places to the left).

$$5.2 \times 10^{-3} = 0.0052$$

Try some yourself (4.3.1)

Solutions on page 108.

- 1 Express each of the following numbers in scientific notation.
 - (a) Light travels 9460 700 000 000 km in a year.
 - (b) The average distance from the centre of the Earth to the centre of the Moon is 384 400 km.
 - (c) Saturn is 1427 000 000 km from the Sun.
 - (d) The mass of the Earth is 5976 000 000 000 000 000 000 kg.
 - (e) The Mediterranean Sea has an area of 2504 000 km².
 - (f) Annapurna 1 is 8075 m high.
- 2 The following numbers are given in scientific notation. Write each out as a single decimal number.
 - (a) Ben Nevis is 1.343×10^3 m high.
 - (b) Mercury is 5.8×10^7 km from the Sun.
 - (c) The distance from the Earth to the nearest star is about 4.3×10^{13} km.
 - (d) The Earth has a circumference of 4.0078×10^4 km at the equator.
 - (e) The span of the Golden Gate Bridge is 1.28×10^3 m.
 - (f) The Arctic Ocean has an area of 1.3986×10^7 km².
- 3 Express each of the following in scientific notation.
 - (a) 0.53 (b) 0.0075 (c) 0.000 004 (d) 0.000 020 01
- 4 Express each of the following in conventional decimal notation.
 - (a) 2.03×10^{-4} (b) 1.4×10^{-3} (c) 5.67×10^{-5}

4.3.2 Using scientific notation

Scientific notation can be very useful when estimating the answers to calculations involving very large and/or small decimal numbers.

Example 9

A lottery winner won £7851 000. He put the money straight into a deposit account which earns 7.5% interest per annum (i.e. each year). If he wanted to live off this interest, how much per day would it be?

Solution

The amount is $£7851\,000 \times \frac{7.5}{100} \div 365$.

Estimate first to provide a check for the calculator work.

$$7851\,000 = 7.851 \times 10^6 \simeq 8 \times 10^6$$

$$\frac{7.5}{100} = 7.5 \times 10^{-2} \simeq 8 \times 10^{-2}$$

$$365 = 3.65 \times 10^2 \simeq 4 \times 10^2$$

So the estimate becomes:

$$(8 \times 10^6) \times (8 \times 10^{-2}) \div (4 \times 10^2).$$

Now separate the digits from the powers of 10 to give:

$$(8 \times 8 \div 4) \times (10^6 \times 10^{-2} \div 10^2).$$

Since $8 \times 8 \div 4 = 16$ and $10^6 \times 10^{-2} \div 10^2 = 10^{6+(-2)-2} = 10^2$, the estimate is $16 \times 10^2 = 1600$ (£1600 a day!).

On a calculator, $7851\,000 \times 0.075 \div 365$ gives £1613 rounded to the nearest pound, which is quite close to the estimate.

Solutions on page 109.

Try some yourself (4.3.2)



- 1 Use the method outlined in Example 9 to estimate each of the following, and then use your calculator to evaluate each, giving your answers to six significant figures.

(a) $2521 \div 38$ (b) 17.85×286.3 (c) $1452 \div 0.0072$

(d) $0.0053 \times 0.0078 \div 0.6821$

Outcomes

Now that you have studied Module 4 you should be able to:

- ◇ evaluate the squares, cubes and other powers of positive and negative numbers with or without your calculator;
- ◇ estimate square roots and calculate them using your calculator;
- ◇ describe the power notation for expressing numbers;
- ◇ use your calculator to find powers of numbers;
- ◇ multiply and divide powers of the same number;
- ◇ understand and apply negative powers, the power 0 and the power 1 for any number;
- ◇ express any given number in scientific notation;
- ◇ convert a number from scientific notation to conventional decimal form;
- ◇ use scientific notation and the rules of powers to find estimates for calculations;
- ◇ read results displayed in scientific notation on your calculator.

Solutions

Section 1.1.1

1 The numbers in digital form are:

808 000 2000 024 9998

So, in ascending order, we have:

9998, 808 000, 2000 024.

2 $2490 \times 100 = 249\,000$

Notice that you have to add two *extra* zeros.

Section 1.1.2

1 $370.76 \div 1000 = 0.370\,76$.

So $370.769\text{ g} = 0.370\,76\text{ kg}$.

2 In ascending order (in metres):

0.0714, 0.0741, 0.1074, 0.1704.

Section 1.1.3

1

(a) $\frac{1}{4}$ (b) $\frac{7}{12}$ (c) $\frac{3}{7}$

2

(a) Since $\frac{4}{6} = \frac{2 \times 2}{3 \times 2}$ and $\frac{12}{18} = \frac{2 \times 6}{3 \times 6}$, both $\frac{4}{6}$ and $\frac{12}{18}$ are equivalent fractions for $\frac{2}{3}$.

Neither $\frac{3}{4}$ nor $\frac{7}{10}$ is equivalent to $\frac{2}{3}$.

(b) (i) $\frac{5}{8}$ and $\frac{10}{16}$ are equivalent fractions. $\frac{5}{8}$ is in its simplest form.

(ii) $\frac{3}{4}$ and $\frac{5}{6}$ are not equivalent fractions.

(iii) $\frac{12}{21}$ and $\frac{4}{7}$ are equivalent fractions. $\frac{4}{7}$ is in its simplest form.

(iv) $\frac{5}{100}$ and $\frac{1}{20}$ are equivalent. $\frac{1}{20}$ is the simplest.

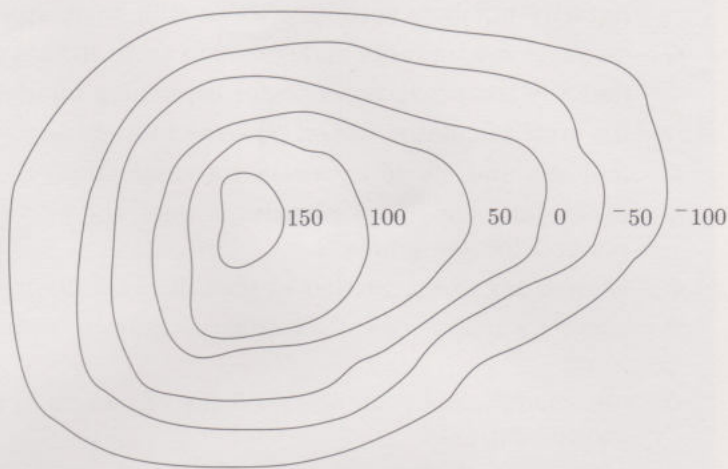
(v) $\frac{1}{3}$ and $\frac{3}{5}$ are not equivalent.

Section 1.1.4

1

(a)

2



Section 1.2.1

1

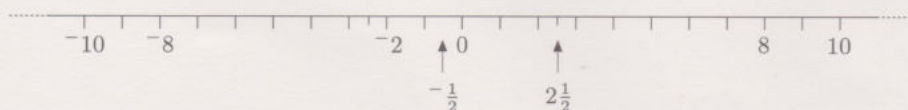
- (a) The age of a young kitten would be measured in weeks or days.
- (b) The distance between train stations could be measured in kilometres (or miles).
- (c) The active ingredients of a pill are usually measured in milligrams.
- (d) The cost of leaving a light on for 24 hours will vary according to the type of bulb and the price of electricity, but it is likely to be measured in pence.

Section 1.2.2

1 1.65 m is 165 cm.

2

- (a) To convert millimetres to centimetres, divide by 10. So 675 mm is 67.5 cm.
- (b) 45.2 km is 45 200 m (multiply by 1000).
- (c) 3.5 kg is 3500 g (multiply by 1000).
- (d) 167.2 mm is 0.1672 m (divide by 1000).



(b) A is $-1\frac{3}{5}$ (or -1.6), B is $-\frac{3}{5}$ (or -0.6), C is $1\frac{1}{5}$ (or 1.2).

Section 1.2.3

1

- (a) First convert to centilitres, say:

X is 50 cl; Y is 490 cl; Z is 51 cl.

So the order is X, Z, Y.

- (b) First convert to pence, say:

X is 20 p; Y is 2 p.

So the order is Y, X.

- (c) First convert to grams, say:

X is 39 g; Y is 4 g; Z is 30 g.

So the order is Y, Z, X.

Section 1.3.1

1

$$\begin{array}{r} \text{(a)} \quad 1000\,000 \\ - \quad \quad 300 \\ \hline 999\,700 \end{array}$$

So £999 700 remains.

$$\begin{array}{r} \text{(b)} \quad 25 \\ + \quad 1029 \\ \hline 1054 \end{array}$$

The total is £1054.

- (c)
- $27\,000 \div 9 = 3000$
- They get £3000 each.

- (d)
- $\frac{27\,000}{900} = \frac{270}{9} = 30$
- They get £30 each.

- 2 Convert all the times to days, say:

four weeks is $4 \times 7 = 28$ days;

29 days is already in days.

To convert 360 hours to days, divide by 24.

Writing it as $\frac{360}{24}$ you may spot that you can divide top and bottom by 12 to give $\frac{30}{2} = 15$.
(Or you can divide 360 by 12 and then by 2 to give the same result as dividing by 24.)

Alternatively you may say 10 days is 240 hours, 5 days is 120 hours, so 15 days is 360 hours.)

So, in order:

360 hours, 4 weeks, 29 days.

3

- (a) Multiplying by 1 gives an answer the same as the original number. (There are many possible ways of wording this. You may have said it 'leaves the number unchanged' or 'has no effect'.)

- (b) Multiplying by 0 always gives 0.

Section 1.3.2

1 When adding two numbers together, or multiplying two numbers together, it makes no difference which number you write first. You get the same answer either way round.

e.g. $3 + 4 = 4 + 3$, and $3 \times 4 = 4 \times 3$.

When subtracting one number from another or dividing one number by another, it does matter which number you write first. You get different answers the other way round.

e.g. $8 - 4 = 4$ is not the same as $4 - 8 = -4$;and $8 \div 4 = 2$ is not the same as $4 \div 8 = \frac{1}{2}$.

2

(a) $3 \times (60 + 70) = 3 \times 130 = 390$

(b) $(3 \times 60) + 70 = 180 + 70 = 250$

(c) $(70 - 60) \div 5 = 10 \div 5 = 2$

(d) $70 + (210 \div 30) = 70 + \frac{210}{30} = 70 + 7 = 77$

3

(a) $3 \times 60 + 70 = (3 \times 60) + 70 = 180 + 70 = 250$

(b) $10 - 15 \div 5 = 10 - (15 \div 5) = 10 - 3 = 7$

(c) $20 - 2 \times 8 = 20 - (2 \times 8) = 20 - 16 = 4$

(d) $3 + 16 - 10 = (3 + 16) - 10 = 19 - 10 = 9$

(e) $3 \times 10 \div 5 = (3 \times 10) \div 5 = 30 \div 5 = 6$

4 The order makes a difference in calculations (a), (b) and (c) of Question 3.

5

(a) $\frac{20 + 4}{3 + 5} = \frac{(20 + 4)}{(3 + 5)} = \frac{24}{8} = 24 \div 8 = 3$.

- (b) No, because there are *implicit* brackets around each of the top and bottom line of the fraction. When you write one number over another, you should treat the expressions above and below the line as if they were each in brackets. So the fact that the additions in part (a) were carried out before the division does not represent an exception to the 'division before addition' rule; it simply follows the 'brackets before division' rule.

Section 1.3.3

1

$$(a) \begin{array}{r} 100.001 \\ + 10.100 \\ \hline 110.101 \end{array}$$

$$(b) \begin{array}{r} 100.001 \\ - 10.100 \\ \hline 89.901 \end{array}$$

$$(c) 75.6 \div 0.6 = \frac{75.6}{0.6} = \frac{756}{6} = 126$$

$$(d) 75.6 \times 0.6 = 75.6 \times \frac{6}{10} = 7.56 \times 6 = 45.36$$

(e) Carrying out the division first, use the result of part (c) to give

$$100.001 + 75.6 \div 0.6 = 100.001 + 126 = 226.001.$$

(f) Carry out the addition in brackets first and use the result of part (a) to give

$$\begin{aligned} (100.001 + 10.1) \times 60 &= 110.101 \times 60 \\ &= 1101.01 \times 6 \\ &= 6606.06. \end{aligned}$$

2 $10 \div 3$ gives

$$3 \overline{) 10.101010 \dots} \text{ and so on.}$$

Each time you divide by 3 you get a remainder 1, which gives 10 in the next column, so the decimal carries on forever. This is called a *recurring decimal*, because the 3 recurs repeatedly.

The course calculator gives 3.33333333 (it rounds to 10 digits).

Section 1.3.4

1

$$(a) \frac{3}{10} + \frac{5}{10} = \frac{3+5}{10} = \frac{8}{10} \text{ (or } \frac{4}{5} \text{)}$$

$$(b) \frac{11}{32} - \frac{7}{32} = \frac{11-7}{32} = \frac{4}{32} \text{ (or } \frac{1}{8} \text{)}$$

$$(c) \frac{7}{16} - \frac{2}{16} = \frac{7-2}{16} = \frac{5}{16}$$

$$(d) \frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} \text{ (or } \frac{1}{2} \text{)}$$

$$(e) \frac{1}{3} + \frac{1}{5} = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}.$$

2 There are obviously many different ways of explaining this. For example one student said:

‘First look at the bottom number of each fraction. If these are the same, then this is also the bottom number of the answer. To get the top number in the answer, add the two top numbers in the given fractions.

‘If the bottom numbers are different, multiply the **top** and **bottom** of the first fraction by the **bottom number** of the second. Then multiply the **top** and **bottom** of the second fraction by the **bottom number** of the **first**. Now the two fractions have the **same bottom number** so they can be added as above.’

Your wording may have been very different but it should have given instructions for the case when the denominators are the same and when they are not. Note that the method described here will always work, when bottom numbers are different, although in some cases (e.g. 1(d) above) a quicker method is possible.

3

$$(a) \frac{11}{42} \times \frac{3}{11} = \frac{1}{4} \text{ (Cancel by 3 and 11)}$$

$$(b) \frac{15}{26} \times \frac{9}{10} = \frac{3}{4} \text{ (Cancel by 5 and 3)}$$

$$(c) \frac{3}{4} \text{ of } 2 \text{ means } \frac{3}{4} \times 2 = \frac{3}{2} \times \frac{2}{1} = \frac{3}{1} \text{ (or } 1\frac{1}{2} \text{)}$$

$$(d) \frac{4}{5} \text{ of } \frac{1}{2} \text{ means } \frac{4}{5} \times \frac{1}{2} = \frac{2}{5} \times \frac{1}{1} = \frac{2}{5}$$

4 You want $\frac{3}{8}$ of 720, or $\frac{3}{8} \times 720 = 270$. So there are 270 girls in the school.

Alternatively $\frac{1}{8}$ of 720 is 90. So $\frac{3}{8}$ of 720 = $3 \times 90 = 270$.

5

$$(a) 10 \div \frac{1}{2} = 10 \times 2 = 20 \text{ (reciprocal of } \frac{1}{2} \text{ is } \frac{2}{1} \text{)}$$

$$(b) \frac{2}{5} \div \frac{5}{7} = \frac{2}{5} \times \frac{7}{5} = \frac{14}{25} \text{ (reciprocal of } \frac{5}{7} \text{ is } \frac{7}{5} \text{)}$$

$$(c) 1\frac{1}{3} = \frac{4}{3}. \text{ So } 1\frac{1}{3} \div 2 = \frac{4}{3} \div \frac{2}{1} = \frac{4}{3} \times \frac{1}{2} = \frac{2}{3}.$$

6 There are several ways of explaining this. For example:

‘To multiply two fractions, multiply the top two numbers together (the numerators) and multiply the bottom two numbers together (the denominators). To give the answer in its simplest form, check whether top and bottom numbers have a factor in common and, if so, divide through by it.’

Or

‘To multiply two fractions by cancelling, find a number that goes into (divides) a number on the top and a number on the bottom i.e. cancel. Repeat this until there are no numbers left which will cancel with each other. Then multiply together the top two numbers (the numerators) and the bottom two numbers (the denominators). This fraction should be in its simplest form.’

Section 1.3.5

1

- (a) $-4 - 6 = -10$. If you have an overdraft of £4 and take out £6 you have an overdraft of £10.
- (b) $3 + -5 = 3 - 5 = -2$. Thomas' bank has £3 plus a £5 IOU is equivalent to a £2 IOU.
- (c) $-4 + -7 = -4 - 7 = -11$. A debit of £4 plus a debit of £7 is equivalent to a debit of £11.
- (d) $-5 + 8 = 3$. An overdraft of £5 plus a deposit of £8 results in a balance of £3.
- (e) $4 - -2 = 4 + 2 = 6$. A gift of £4 plus taking away a £2 debt is equivalent to a gift of £6.
- (f) $-3 - -5 = -3 + 5 = 2$. Acquiring a debt of £3 and taking away a debt of £5 results in being £2 better off.

2 Kim started at -37 m and ended at 42 m. You want $42 - -37 = 42 + 37 = 79$. So Kim climbed 79 m.

3 Your list should look something like this.

Dividing a positive number by a positive number gives a positive answer. $4 \div 2 = 2$.

Dividing a negative number by a positive number gives a negative answer.
 $-4 \div 2 = -2$.

Dividing a positive number by a negative number gives a negative answer.
 $4 \div -2 = -2$.

Dividing a negative number by a negative number gives a positive answer.
 $-4 \div -2 = 2$.

4

- (a) $4 \times -3 = -12$
 Four debts of £3 is a debt of £12.
- (b) $-2 \times -7 = 14$
 Thomas' mother taking away two £7 IOUs is equivalent to giving him £14.
- (c) $-3 \times 5 = -15$
 Thomas gives away 3 five pound notes which is equivalent to giving away £15.
- (d) $24 \div -4 = -6$
 Thomas has lots of £4 IOUs. A gift of £24 will subtract 6 of them.

(e) $-40 \div -4 = 10$

An IOU of £40 divided into smaller IOUs of £4 each (which could be paid off more easily from Thomas's pocket money) gives 10 smaller IOUs.

(f) $-45 \div 15 = -3$

A debt of £45 shared between 15 people means each owes £3.

Section 2.1.1

1

Number	Rounded to nearest		
	10	100	1000
325 089	325 090	325 100	325 000
45 982	45 980	46 000	46 000
11 985	11 990	12 000	12 000

2 The corrected entries are in bold.

Number	Rounded to nearest		
	10	100	1000
254 987	254 990	255 000	255 000
12 345	12 350	12 300	12 000
963	960	1 000	1 000

3 7000 000

4

- (a) Day: $68 \times 60 \times 24 = 97\,920$, i.e. about 100 000 beats per day (rounded to nearest hundred thousand).
- (b) Week: $97\,920 \times 7 = 685\,440$, i.e. about 700 000 beats per week (rounded to nearest hundred thousand).

Month: $97\,920 \times 30 = 2\,737\,600$, i.e. about 3 000 000 to nearest million.

Year: because the numbers now are so big it would be appropriate to use numbers that are approximations, e.g.

3 million $\times 12 = 36$ million $\simeq 40$ million.

- (c) Lifetime: A lifetime could last from a very short time to over a hundred years, so an assumption must be made. Supposing it to be 80 years gives
 $40\,000\,000 \times 80 = 3\,200\,000\,000$, i.e. approximately 3 billion.

Your interpretation as if to your friend might be something like:

Well Chris, if your heart beats at about 70 beats per minute, this means that it

beats about a hundred thousand times a day, about seven hundred thousand times a week, about three million a month and about thirty six million a year. So if you live to be 80 you would expect your heart to beat about three billion times. No wonder we should be looking after our hearts!

Section 2.1.2

1

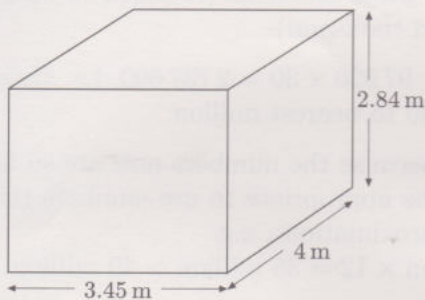
Number	Rounded to		
	1 d.p.	2 d.p.	3 d.p.
0.47265	0.5	0.47	0.473
13.959943	14.0	13.96	13.960
56.09827	56.1	56.10	56.098

Notice the use of zero in the last place in some of the answers. This is necessary to show that the value has been rounded to that number of decimal places.

2 The corrected entries are in bold.

Number	Rounded to		
	1 d.p.	2 d.p.	13 d.p.
3.1415926	3.1	3.14	3.142
22/7	3.1	3.14	3.143
0.019999	0.0	0.02	0.020

3



Rounding to approximate accuracy and then calculating the volume of the tank by using the formula length×width×height gives a volume of approximately $4 \times 3 \times 3 = 36$ cubic metres. 1 cubic metre contains 1000 litres, so 36 cubic metres contains $36 \times 1000 = 36\,000$ litres.

If you need 2 litres per day of water then you can survive $36\,000 \div 2 = 18\,000$ days before it rains. This is about 20 000 days and taking a

year to be about 400 days means that you can survive about $20\,000 \div 400 = 50$ years.

The accurate answer would give 53.69 years (to 2 d.p.). You might not be too concerned to be told that you had only 50 years rather than 53.69 years to live on your deserted island if it did not rain!

Section 2.1.3

1

Number	Round to			
	1 s.f.	2 s.f.	3 s.f.	4 s.f.
2098 765	2 000 000	2 100 000	2 100 000	2 099 000
0.042 6395	0.04	0.043	0.0426	0.04264
0.000 289 99	0.0003	0.00029	0.000290	0.0002900

2

Number	Round to			
	nearest whole number	1 d.p.	1 s.f.	3 s.f.
238.483	238	238.5	200	238
0.763 48	1	0.8	0.8	0.763
0.088 8456	0	0.1	0.09	0.0888

3

(a) £0.048654 is £0.05, to 1 s.f.

So an item costing ASh 278 will be about $278 \times 0.05 = \text{£}13.90$, or about £14.

If you rounded the cost to 1 s.f. too, you would have estimated $300 \times 0.05 = \text{£}15$.

(b) £0.048654 is £0.049, to 2 s.f.

So an item costing ASh 278 will be about $278 \times 0.049 = \text{£}13.62$ (to 2 d.p.).

If you rounded the cost to 2 s.f. too, you would have estimated $280 \times 0.049 \text{ g} = \text{£}13.72$.

4

(a) 580 000 000 (to 2 s.f.), which is five hundred and eighty million miles.

(b) 580 000 000 miles in 8760 hours is $580\,000\,000 \div 8760 = 66\,000$ mph (to 2 s.f.)

Section 2.2

1

- (a) Round the numbers to
 $70 + 20 + 50 + 90 + 90 + 100 = 420$ pence or £4.20. So the bill should be a bit more than £4. (In fact the exact total is £4.16.)
- (b) 784 rounds to 800. So in a year Dave earns about $£800 \times 12$, which is £9600. (In fact he earns exactly £9408 a year.)

2

- (a) Round 27 up to get 30. So the area $\simeq 30 \times 3 = 90$ square metres.
- (b) Round the numbers to get 20×10 . So the area $\simeq 20 \times 10 = 200$ square metres. (Since the exact answer is $19 \times 11 = 209$ square metres, you'd be in a bit of a mess here if you just bought 200 square metres. This example illustrates that in many practical situations it is safer to overestimate than underestimate.)

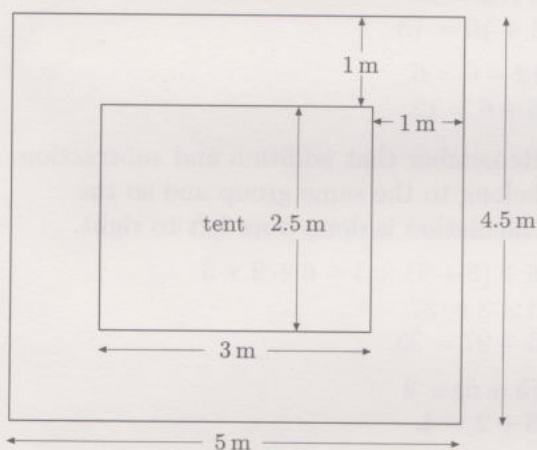
3

- (a) $56\,432 \simeq 60\,000$ (1 s.f.).

If 2 people share a tent then an estimate of the number of tents is

$$60\,000 \div 2 = 30\,000 \text{ tents.}$$

(b)



Area required by each tent with boundary is

$$\begin{aligned} \text{Area} &= \text{Length} \times \text{breadth} \\ &= 5 \text{ m} \times 4.5 \text{ m} \\ &= 22.5 \simeq 20 \text{ square metres.} \end{aligned}$$

So area required is

$$30\,000 \times 20 \simeq 600\,000 \text{ square metres.}$$

- (c) One fifth of people will use the shower, i.e. $60\,000 \times \frac{1}{5} = 12\,000$ people.

Each shower takes 5 minutes, hence the number of showering minutes is

$$12\,000 \times 5 \simeq 60\,000 \text{ minutes.}$$

Number of minutes between 7am and 10am

$$= 3 \times 60 = 180 \text{ minutes} \simeq 200 \text{ minutes.}$$

Number of shower cubicles needed

$$\simeq 60\,000 \div 200 \simeq 300 \text{ cubicles.}$$

- (d) Number of toilet visits is number of people times number of visits per day
 $\simeq 60\,000 \times 4 = 240\,000$ units.
 Number of toilet minutes
 $\simeq 240\,000 \times 2 \simeq 480\,000$ minutes.
 Number of minutes in 24 hrs
 $\simeq 24 \times 60 \simeq 1440 \simeq 1500$ minutes.
 Number of toilet cubicles needed is

$$480\,000 \div 1500 \simeq 320 \text{ toilet cubicles.}$$

- (e) You need to estimate here how many meals per day and how many cans of drink each person may require, e.g. 3 meals and 6 drinks.

So number of meals $\simeq 60\,000 \times 3 \simeq 180\,000$ but since some people may not eat 3 meals per day you might order only 150 000 meals.

Number of drinks $\simeq 60\,000 \times 6 \simeq 360\,000$ but if you want to offer a choice of drinks you might order more, especially if you expect the weather to be very hot so people are very thirsty...

4

- (a) (i) Rough estimate: rounding to the nearest thousand:

$$75\,450 \simeq 75\,000, \quad 27\,654 \simeq 28\,000.$$

$$\begin{aligned} \text{So } 75\,450 - 27\,654 &\simeq 75\,000 - 28\,000 \\ &= 47\,000. \end{aligned}$$

Or you may have rounded to the nearest 10 000, to give
 $80\,000 - 30\,000 = 50\,000$.

The exact value is 47 796.

- (ii) Rough estimate: rounding to one significant figure (i.e. to the nearest 100 000):

$$676\,295 \simeq 700\,000, \quad 984\,122 \simeq 1\,000\,000$$

$$\begin{aligned} \text{So } 676\,295 + 984\,122 &\simeq 700\,000 + 1\,000\,000 \\ &= 1\,700\,000. \end{aligned}$$

The exact value is 1 660 417.

- (iii) $16.859 \times 4.962 \simeq 17 \times 5 = 85$.

The exact value is 83.654 358.

- (iv) $3.512 \div 720.5 \simeq 4 \div 700$. This may not be very helpful. Another approach is to spot that, if you round 3.512 to 3.5, you can

divide top and bottom by 7:

$$\frac{3.5}{700} = \frac{0.5}{100} = 0.005.$$

(Don't worry if you didn't spot this. It doesn't matter as long as you found a rough estimate by some means.)

The calculator gives 0.004 874 3928 (to eight significant figures).

- (b) (i) Wrong. $631 - 529 \simeq 600 - 500 = 100$, not 6000.
 (ii) Right.
 $13\,726 + 4317 \simeq 14\,000 + 4\,000 \simeq 18\,000$, so it could be right.
 (iii) Wrong.
 $1066 \times 45 \simeq 1000 \times 50 = 50\,000$, not 7000.
 (iv) Wrong.
 $1066 \div 0.5 \simeq \frac{1000}{0.5} = \frac{10\,000}{5} = 2000$, not 20.

One way of noticing there must be a mistake is to notice that dividing by a decimal number between 0 and 1 always gives an answer *larger* than the original number.

5

- (a) Estimate: $0.086 + 219.2 \simeq 0 + 220 = 220$.
 The calculator gives 219.286.
 (b) $0.050 - 0.048 \simeq 0.05 - 0.05 = 0$. This is not good enough—all significant figures are lost! You cannot estimate this roughly—use all the given figures: $0.050 - 0.048 = 0.002$. There is only one significant figure in the answer, so you cannot round it to two significant figures.
 (c) $457.3 - 64.5 + 302.4 \simeq 460 - 60 + 300 = 400 + 300 = 700$.
 The calculator gives $695.2 \simeq 700$, to 2 s.f. (Notice that, although there only appears to be one significant figure here, the answer was rounded to *two*; both the 7 and the first 0 are significant.)
 (d) $441.7 \times 5.2 \simeq 400 \times 5 = 2000$.
 The calculator gives $2296.84 \simeq 2300$, to 2 s.f.
 (e) $53.4 \times 70.9 \div 22.2 \simeq 50 \times 70 \div 20 = \frac{3500}{20} = \frac{350}{2} = 175$.

The calculator gives $170.543\,2432 \simeq 170$, to 2 s.f.

$$(f) \quad 217.5 + 60.3 \times 17.7 \simeq 200 + 60 \times 20 = 200 + 1200 = 1400.$$

The calculator gives $1284.81 \simeq 1300$, to 2 s.f.

$$(g) \quad (1285 - 329) \times 0.023 \simeq (1300 - 300) \times 0.02 = 1000 \times 0.02 = 1000 \times \frac{2}{100} = 20.$$

The calculator gives $21.988 \simeq 22$, to 2 s.f.

Section 2.3.1

1 No. You owe £8.70, but the coupons have paid £3.50, so you owe $£8.70 - £3.50 = £5.20$. The assistant wrongly added instead of subtracting.

2 No. Tom should have added not subtracted the value of the discount coupons, since they are the equivalent of money received, not spent. Alternatively he could have subtracted them from the supermarket bill before subtracting this: $15 - (12.50 - 1.50)$ is the same as $15 - 12.50 + 1.50$.

Section 2.3.2

1

- (a) $5 \times 2 = 10$
 $3 + 10 = 13$.
 (b) $12 - 6 = 6$
 $6 + 6 = 12$.

Remember that addition and subtraction belong to the same group and so the calculation is done from left to right.

- (c) $6 + (5 + 4) \times 3 = 6 + 9 \times 3$
 $9 \times 3 = 27$
 $6 + 27 = 33$.
 (d) $(3 + 5) \div 2$
 $8 \div 2 = 4$.
 (e) $12 - (6 + 6) = 12 - 12 = 0$.
 (f) $6 + 5 + 4 \times 3 \div 2 = 6 + 5 + 6 = 17$.

Section 2.3.3

1 He has not used the same units. The figure 10 he used refers to feet, so in a calculation using yards he needs $\frac{10}{3}$ (3 feet = 1 yard). The cost should be

$$£25.50 \times 6 \times \frac{10}{3} = £510.$$

2 The errors are

1. Not converting the centimetres to metres.
2. The order of operations for calculating the end door panels.

Correct answer should be

Areas are:

sides $2 \times 3 \times 2.5 = 15$ metres squared

doors $2 \times (1.5 + 1.5) = 6$ metres squared

Total area = 21 metres squared

Number of litres = $21 \div 5 = 4.2$ litres

Therefore Kim needs to buy 5 litres of proofing agent.

Section 2.3.4

1 You may have worked out a ball-park figure rather than under and over estimates or you may have used different estimates. This does not matter as long as you can check on whether your calculator answer is what you would expect.

- Overestimate: $24 \div 4 = 6$
Underestimate: $20 \div 5 = 4$.
ANS = 5.368932039.
- Overestimate: $1 \times 11 = 11$
Underestimate: $0.8 \times 10 = 8$.
ANS = 8.97309465.
- Overestimate: $1010 \times 24 \times 300$
 $= 1010 \times 7200 = 7\,272\,000$
Underestimate: $1000 \times 20 \times 270 = 5\,400\,000$
ANS = 6696385.117
- Overestimate: $(13 + 15 + 64) \div 10 = 9.2$
Underestimate: $(12 + 14 + 63) \div 13$
 $= 89 \div 13 \simeq 7$
ANS = 7.252631579.

2 Rough estimate = $7 \times 5 \times 0.2 \simeq 7$ cubic metres.

Al's estimate = 61 cubic metres which seems rather large, by a factor of about ten, so you should not be happy with his estimate.

He has said that 200 mm = 2 m whereas he should have used 0.2 m: he should have divided 200 by 1000 rather than 100. (There are 1000 mm in 1 m.)

3 £5699.7 seems a lot for a stair carpet.

Estimate of answer: Length needed for one step $\simeq 0.2 + 0.2 \simeq 0.4$ m.

Length needed for 10 stairs $\simeq 10 \times 0.4 = 4$ m

So length needed for 30 stairs $\simeq 4 \times 3 = 12$ m.

Landings need $\simeq 1 + 6 \simeq 7$ m

So total length needed = $12 + 7 = 19 \simeq 20$ m.

So cost would be about $20 \times 5 = £100$ which is a lot less than £5699.70.

The stairs are measured in cms and the landings are measured in metres. So in calculating 'total length' your friend has added numbers of different units. The cost of the carpet is in pounds per metre, so it is sensible to convert centimetres to metres for the calculation.

Measuring in metres gives $6.6 + \frac{1260}{100} = 19.2$ metres. Your friend might need a safety margin so he should buy 20 metres at a cost of $20 \times £4.50 = £90$.

Section 2.3.5

1

Context	Appropriate rounding
Carpet floor area = 26.456 sq metres	27 sq metres (Round up if you are buying carpet)
Interest earned = £109.8765439	£109.88 (Money is given to 2 d.p.)
Bill for £84.90 shared by 7 people	£12.13 (Round up to have enough money)
Length of room = 5.3876956 m	5.39 m (Rounded to nearest cm – depends on context)
Estimated attendance = 24.678	25 (Cannot have part of a person)
Thickness of paper = 0.0543 mm	0.05 mm (cannot visualize more accuracy)

2 You need to round up to cover the cost of the coach, and you need to round to the nearest penny at least. So $£85 \div 6 \simeq £14.17$ or even £15 (including a tip for the driver).

3 You would probably want to charge a price that can easily be handled by buyers and sellers at the sale, and so rounding to the nearest pound or ten pence would be sensible, giving £5.00 or £5.10. However, a reduction to £4.99 might attract more buyers.

4 Your friend would have been better converting all the lengths to metres first, as there is an error in the total. You might have spotted this since 7.04 m plus two other lengths must give more than 3.652 m. There is also another error: 1 m 76 mm is 1.076 m; and 4 times this is 4.304 m.

An estimate of the total lengths would have suggested something was wrong. Nine curtains at between half a metre and a metre in length will need between $4\frac{1}{2}$ and 9 metres. At £3 per metre this is well over £11.

Your friend also needs to round up the lengths of material to allow for hemming.

So the corrected calculation would be like the one below:

window 1: 2 curtains length 0.670 m	= 1.340 m
window 2: 4 curtains length 1.076 m	= 4.304 m
window 3: 3 curtains length 0.536 m	= 1.608 m
total length	= 7.252 m

Round up the length to 7.5 or 8 m to allow for hems, etc.

Cost at £3 per metre:

7.5 m costs £22.50; 8 m costs £24.00.

(You might also have queried his measurements, since some windows seem very small. Might he have confused centimetres with inches? For example, should the first length really be 67 in?)

5 $200 \div 35 = 5.714285714$.

Round down to the nearest whole card, so that there are sufficient cards to go round. So deal 5 cards each.

Section 3.1.1

1 70 is 5×14 , so

$$\frac{\text{distance in miles}}{\text{distance in kilometres}} = \frac{5}{8} = \frac{5 \times 14}{8 \times 14} = \frac{70}{112}.$$

70 miles is about the same as 112 kilometres.

Alternatively $1 \text{ km} = \frac{5}{8} \text{ miles}$. So $110 \text{ km} = 110 \times \frac{5}{8} \text{ miles} = 68.75 \text{ miles}$.

So the two speed limits are very close but with 70 miles per hour being the higher.

2 In order to compare the prices of the two cereal packs it is best to work out the price per gram for each.

The large packet will cost £1.85 for 450 g. This is $\frac{185}{450}$ or 0.411p per gram.

The New extra large packet will cost £2.35 for 625 g. This is $\frac{235}{625}$ or 0.376p per gram. Obviously the new extra large pack is the better bargain.

(But the extra large package is 37 cm tall (about 13"). Unluckily it might not fit in the kitchen cupboard.)

3 5 kg costs £2.70. You want to know, at this rate, how much you get for 75 p. First work out how much you get for £1:

if £2.70 buys 5 kg then £1 buys $\frac{5}{2.7}$ kg.

So 75 p should buy $0.75 \times \frac{5}{2.7} \simeq 1.39 \text{ kg}$.

If three potatoes weigh 1.39 kg each, one weighs about 0.46 kg. So each potato in the pack would need to weigh around half a kilogram for the pack to be a better bargain.

Baking potatoes are big, but it's unlikely that a pack of three baking potatoes would weigh $1\frac{1}{2} \text{ kg}$.

Section 3.1.2

1

(a) $\frac{1}{5} = 0.2$

(b) $\frac{1}{3} = 0.3333 \dots$ this repeats indefinitely

It can be written as $0.\dot{3}$, where the dot over the 3 means that it is repeated. It is pronounced 'nought point three recurring'. As a decimal, you need to terminate after some finite number of places, e.g. 0.3333 to four decimal places.

(c) $1\frac{1}{2} = \frac{3}{2} = 1.5$

(d) $2\frac{1}{4} = \frac{9}{4} = 2.25$

Alternatively, 2 is 2.00, $\frac{1}{4}$ is 0.25, and adding them together gives 2.25.

2

(a) $0.25 = \frac{25}{100} = \frac{1}{4}$

(b) $1.135 = \frac{1135}{1000} = \frac{227}{200}$

(c) $0.064 = \frac{64}{1000} = \frac{8}{125}$

Section 3.1.3

1 There are at least three ways of answering this:

- (a) 70 seconds is $\frac{70}{60} = \frac{7}{6} = 1.1666\dots$ minutes, which is less than 1.2 minutes.
- (b) 1.2 minutes is $1\frac{2}{10} = 1\frac{1}{5}$ minutes. 70 seconds is 1 minute 10 seconds, i.e. $1\frac{10}{60} = 1\frac{1}{6}$ minutes. Since $\frac{1}{6}$ is less than $\frac{1}{5}$, 70 seconds is less than 1.2 minutes.
- (c) 1.2 minutes is $1.2 \times 60 = 72$ seconds, which is greater than 70 seconds.

So 1.2 minutes is greater than 70 seconds.

2

- (a) Speed is the ratio of distance travelled to time taken, which in the cheetah's case is 400 metres to 15 seconds. So its speed is $\frac{400}{15} \simeq 27$ metres per second (to 2 s.f.).

To determine this speed in kilometres per hour, both units need changing.

400 metres are $\frac{400}{1000}$ kilometres

15 seconds are $\frac{15}{3600}$ hours.

So $\frac{\text{distance}}{\text{time}}$ in the new units is

$$\begin{aligned} \frac{400}{1000} \div \frac{15}{3600} &= \frac{400}{1000} \times \frac{3600}{15} \\ &= \frac{8400}{1000} \times \frac{3600^{12}}{15} \\ &= 8 \times 12 \\ &= 96 \text{ kilometres per hour} \end{aligned}$$

So the cheetah's speed is 96 kilometres per hour (over a distance of 400 metres).

- (b) In 16 hours the sloth moves 1 mile. It will take $5 \times 16 = 80$ hours to move 5 miles which is 8 km. So it will take 80 hours to move 8 km. In 1 hour it will move $8/80$ km or 0.1 km. So the sloth will move at a speed of 0.1 km/h (assuming it does not fall asleep!).

To convert this speed to metres per second,

0.1 km is 100 metres

1 hour is 3600 seconds

So speed is $\frac{100}{3600} \simeq 0.028$ metres per second (to 2 s.f.)

3 As a fraction of an hour, 11 minutes is $\frac{11}{60}$ hours. To convert this to a decimal, divide 11 by 60 to get 0.18333... This is greater than 0.17, so 11 minutes is longer. (Alternatively, 0.17 hours is $60 \times 0.17 = 10.2$ minutes, so 11 minutes is longer.)

Section 3.2

1 If it takes 5 hours to cook $1\frac{1}{2}$ kg then the time to cook 1 kg is $5 \div 1\frac{1}{2}$ hours.

$$5 \div \frac{3}{2} = 5 \times \frac{2}{3} = \frac{10}{3} = 3\frac{1}{3} \text{ hours.}$$

So a 2 kg pudding takes $2 \times 3\frac{1}{3} = 6\frac{2}{3}$ hours (6 hours 40 mins), assuming that cooking time is proportional to weight.

2 C: 4 programmers take 1 year.

So 1 programmer would take 4 years.

3 programmers would take $\frac{4}{3}$ years = $1\frac{1}{3}$ years or 1 year 4 months.

3

- (a) First work out how long it would last one person, remembering that it will last *longer* for one person:

it lasts 1 week for 7 people, so it lasts 7 weeks for 1 person.

It will last less than this for 2 people—you need to divide by 2: i.e. it lasts $\frac{7}{2} = 3\frac{1}{2}$ weeks for 2 people.

- (b) You know that in $3\frac{1}{2}$ weeks 2 people get through 10 kg of potatoes, so in 1 week 2 people get through $\frac{10}{3\frac{1}{2}} \simeq 2.86$ kg. In practice you would probably buy 3 kg most weeks, and 2 kg when there seemed to be a lot of potatoes left over from the previous week.

Section 3.3.1

1

- (a) (i) $40\% = \frac{40}{100} = \frac{2}{5}$
 (ii) $8\% = \frac{8}{100} = \frac{2}{25}$
 (iii) $70\% = \frac{70}{100} = \frac{7}{10}$
 (iv) $5\frac{1}{2}\% = \frac{11}{2} \times \frac{1}{100} = \frac{11}{200}$
- (b) (i) $50\% = \frac{50}{100} = 0.5$
 (ii) $85\% = \frac{85}{100} = 0.85$
 (iii) $7\% = \frac{7}{100} = 0.07$
 (iv) $17\frac{1}{2}\% = 17.5\% = \frac{17.5}{100} = 0.175$
- (c) (i) $75\% \text{ of } 840 = 0.75 \times 840 = 630$,
 so 630 women were in paid work.
 (ii) $90\% \text{ of } 840 = 0.9 \times 840 = 756$,
 so 756 men were in paid work.
 (iii) $18\% \text{ of } 840 = 0.18 \times 840 = 151.2$. But 151.2 is not a whole number. So 151 couples had a joint income of less than £160 per week. (The survey result must have been rounded, as a percentage, to the nearest whole number.)

2

- (a) (i) $0.8 = 80\%$
 (ii) $0.21 = 21\%$
 (iii) $0.70 = 70\%$
 (iv) $2.4 = 240\%$
- (b) (i) $\frac{1}{2} = 0.5 = 50\%$
 (ii) $\frac{1}{8} = 0.125 = 12.5\% \simeq 13\%$
 (iii) $\frac{1}{9} \simeq 0.1111 \simeq 11\%$
 (iv) $1\frac{1}{7} \simeq 1.143 \simeq 114\%$

3 2100 out of 5000 is

$$\frac{2100}{5000} \times 100\% = 42\%.$$

The swing is $42 - 10 = 32$ percentage points.

Section 3.3.2

1

- (a) $117\frac{1}{2}\% \text{ of } £35.75 = 1.175 \times £35.75$
 $= £42.00625$. In practice this would be rounded to the nearest penny, i.e. £42.01 (or £42.00 perhaps).
- (b) A 10% discount would reduce it to 90% of its previous selling price, i.e.
 $0.9 \times £35.75 = £32.175 \simeq £32.18$.
- (c) If VAT is added first you get
 $0.9 \times £42.01 = £37.809 \simeq £37.81$.
 If the discount is taken first you get
 $1.175 \times £32.18 = £37.8115 \simeq £37.81$.
 Thus it makes no difference to the final price whether you take the discount or the VAT first. The reason for this is, essentially, that the order in which you do successive multiplication doesn't matter:

$$0.9 \times 1.175 \times 35.75 = 1.175 \times 0.9 \times 35.75$$

2 A 12% decrease would reduce it to 88% of the original time, i.e. $0.88 \times 90 = 79.2$ minutes.

$$0.2 \text{ of a minute} = 0.2 \times 60 = 12 \text{ secs.}$$

So 79.2 minutes = 79 minutes and 12 secs.

3 If increase on fares is 7.4% then you would need to find

$$(100\% + 7.4\%) \text{ i.e. } 107.4\% \text{ of } £1260.$$

$$1.074 \times 1260 = 1353.24.$$

So the season ticket will cost £1353.24 after the increase.

4 The London to Geneva fare is increased by 17%. So it will now cost 117% of £165, i.e. $1.17 \times £165 = £193.05$.

The London to New York fare is increased by 15%. So it will now cost 115% of £376, i.e. $1.15 \times £376 = £432.40$.

5

(a) Population after 1 year is likely to be

$$(100\% - 4\%) \text{ of } 4650,$$

$$\text{i.e. } 96\% \text{ of } 4650,$$

$$\text{which is } 0.96 \times 4650 = 4464 \text{ people.}$$

(b) Population after 2 years is likely to be

$$96\% \text{ of } 4464,$$

$$\text{i.e. } 0.96 \times 4464 = 4285.44 \text{ people,}$$

but you can't have 0.44 of a person.

So approximately 4285 people.

Section 4.1.1

1

- (a) $10^2 = 10 \times 10 = 100$
 (b) $100^2 = 100 \times 100 = 10\,000$
 (c) $0.1^2 = 0.1 \times 0.1 = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100} = 0.01$
 (d) $0.01^2 = 0.01 \times 0.01 = \frac{1}{100} \times \frac{1}{100} = \frac{1}{10\,000} = 0.0001$
 (e) $\left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
 (f) $(-3)^2 = -3 \times -3 = 9$
 (g) $(-1)^2 = -1 \times -1 = 1$

2 If you write -1^2 it is not clear whether you mean to square 1 first then make the answer negative, giving -1 , or to square -1 , giving 1. The course calculator would assume you meant $-(1)^2$, and return the answer -1 .

3

- (a) Rough check: $100^2 = 10\,000$.
 Calculator gives $96^2 = 9216$.
 (b) Rough check: $10^2 = 100$.
 Calculator gives $10.65^2 = 113.4225$.

Section 4.1.2

1 A square of side 9 m has an area of 81 m^2 . So 80 m^2 would be a little less than this. $\sqrt{80} = 8.94\dots$. So the length of the side of the square lawn would be 8.9 m.

2

- (a) Since $12 \times 12 = 144$, the answer must be $\sqrt{144} = 12$. The calculator confirms this.
 (b) Since $3^2 = 9$ and $4^2 = 16$, $\sqrt{14.4}$ must lie between 3 and 4.
 Calculator gives $\sqrt{14.4} \simeq 3.7947$ rounded to four decimal places.
 (c) Since $1^2 = 1$ and $2^2 = 4$, $\sqrt{1.44}$ must lie between 1 and 2.
 Calculator gives $\sqrt{1.44} = 1.2$.
 (d) Since $0.3^2 = 0.09$ and $0.4^2 = 0.16$, $\sqrt{0.144}$ must lie between 0.3 and 0.4.
 Looking at the pattern of parts (a) and (c), and since $\sqrt{14.4} = 3.7947$, a good guess would be that $\sqrt{0.144} = 0.3795$ rounded to four decimal places.
 Indeed, the calculator gives $\sqrt{0.144} \simeq 0.3795$ rounded to four decimal places.

Section 4.1.3

1 Since a litre is one cubic decimeter, the question is asking how many cubic decimetres are in a cubic metre.

1 metre = 10 decimetres
 so $1\text{ m}^3 = 10^3$ cubic decimetres.

Hence there are 1000 litres in a cubic metre.

2

- (a) $(-1)^3 = -1 \times -1 \times -1 = 1 \times -1 = -1$
 (b) $3^3 = 3 \times 3 \times 3 = 9 \times 3 = 27$
 (c) $100^3 = 100 \times 100 \times 100 = 10\,000 \times 100 = 1\,000\,000$ (a million)
 (d) $(0.1)^3 = 0.1 \times 0.1 \times 0.1 = \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{100} \times \frac{1}{10} = \frac{1}{1000} = 0.001$

3

- (a) Estimate: $(-1)^3 = -1$.
 Calculate: $(-1.2)^3 = -1.728$.
 (b) Estimate: $3^3 = 27$.
 Calculate: $3.3^3 = 35.937$.
 (c) Estimate: $100^3 = 1\,000\,000$.
 Calculate: $(101)^3 = 101^3 = 1\,030\,301$.
 (d) Estimate: $.1^3 = .001$.
 Calculate: $0.121^3 = 0.00177$ (3 s.f.)

4

- (a) $\sqrt[3]{1} = 1$ since $1^3 = 1$.
 (b) $\sqrt[3]{1000} = 10$ since $10^3 = 1000$.

Section 4.2.1

1

- (a) $10^7 = 10\,000\,000$
 (b) $10^8 = 100\,000\,000$
 (c) $3^4 = 3 \times 3 \times 3 \times 3 = 27 \times 3 = 81$
 (d) $(-2)^2 = -2 \times -2 = 4$
 (e) $(-2)^3 = 4 \times -2 = -8$

2

- (a) $11^7 = 19\,487\,171 \simeq 19\,500\,000$ (3 s.f.)
 (b) $10.8^8 = 185\,093\,021 \simeq 185\,000\,000$ (3 s.f.)
 (c) $3.14^4 = 97.21171216 \simeq 97.2$ (3 s.f.)
 (d) $(-2.2)^2 = 4.84$
 (e) $-(2.01)^3 = -8.120601 \simeq -8.12$ (3 s.f.)

3

- (a) There are 12 zeros in 10^{12} . The number of zeros for a positive power of 10 is the same as the power.
- (b) An odd power of a negative number is negative. An even power of a negative number is positive.

4 After 0 months (i.e. at the start) there are 2 hamsters = 2

After 1 month there are 2×2 hamsters i.e.

$$2^2 = 4$$

After 2 months there are $2 \times 2 \times 2$ hamsters i.e.

$$2^3 = 8$$

So after 12 months there are $2^{13} = 8192$ hamsters.

5 $\frac{1}{10}$ of 1000 = 100

So 100 seeds per flower germinate each year.

At 0 years there is 1 dandelion.

After 1 year there are $1 \times 100 = 100$ dandelions (which can be written as 100^1).

After 2 years there are

$$100 \times 100 = 100^2 = 10\,000 \text{ dandelions.}$$

So after 5 years an estimate is

$$100^5 = 10\,000\,000\,000 \text{ dandelions (or 10 billion).}$$

6 On day 1 there are 5 cells

On day 2 there are $5 \times 2 = 10$ cells

On day 3 there are $5 \times 2 \times 2 = 5 \times 2^2 = 20$ cells

On day 4 there are $5 \times 2 \times 2 \times 2 = 5 \times 2^3 = 40$ cells.

So on day 10 there are $5 \times 2^9 = 5 \times 512 = 2560$ cells.

3

(a) $1024^0 = 1$

(b) $1024^1 = 1024$

(c) $5^{-1} = \frac{1}{5^1} = \frac{1}{5} = 0.2$

(d) $10^{-4} = \frac{1}{10^4} = \frac{1}{10\,000} = 0.0001$

4 $10^{-6} = 0.000\,001$; there are five zeros after the decimal point. For any given negative power of 10, say $10^{-(\text{NUMBER})}$ there is one fewer zeros than NUMBER. This works for 10^{-1} , too, since $10^{-1} = 0.1$, which has no zeros after the decimal point, i.e. one fewer than 1.

5

- (a) Take an example, which makes the explanation easier than describing it in the abstract. Consider 10^{-6} . Now -6 is just the same as $0 - 6$, so $10^{-6} = 10^{0-6}$. Using the rule for dividing powers, $10^{0-6} = 10^0 \div 10^6$. But $10^0 = 1$. So $10^{-6} = 1 \div 10^6$. This argument would apply equally well to any number, since any number to the power 0 is 1. (Alternatively you may have started from the other end, for example, by showing that $\frac{1}{10^6} = \frac{10^0}{10^6} = 10^{0-6} = 10^{-6}$.)

(b) (i) $\frac{1}{4} = 4^{-1} = 0.25$

(ii) $\frac{1}{10^2} = 10^{-2} = 0.01$

Section 4.2.2

1

(a) $10^{-2} = 0.01$

(b) $10^2 \times 10^3 = 10^{2+3} = 10^5 = 100\,000$

(c) $10^7 \div 10^4 = 10^{7-4} = 10^3 = 1000$

(d) $10^4 \div 10^7 = 10^{4-7} = 10^{-3} = \frac{1}{1000} = 0.001$

2 Using Question 1 as a rough check, the calculator gives:

(a) $10.3^{-2} = 0.009\,426$ (to 4 s.f.)

(b) $10.1^2 \times 10.11^3 = 105\,400$ (to 4 s.f.)

(c) $10.112^7 \div 10.21^4 = 994.8$ (to 1 d.p.)

(d) $10.12^4 \div 10.351^7 = 0.000824$ (to 3 s.f.)

It is a good idea to get in the habit of estimating your answers, even if not specifically asked to do so.

Section 4.3.1

1

(a) Light travels at 9.4607×10^{12} km in a year.

(b) The distance from the centre of the Earth to the centre of the Moon is 3.844×10^5 km.

(c) Saturn is 1.427×10^9 km from the Sun.

(d) The mass of the Earth is 5.976×10^{24} kg.

(e) The Mediterranean Sea has an area of 2.504×10^6 km².

(f) Annapurna 1 is 8.075×10^3 m high.

2

- (a) Ben Nevis is 1343 m high.
 (b) Mercury is 58 000 000 km from the Sun.
 (c) The distance from the Earth to the nearest star is about 43 000 000 000 000 km.
 (d) The Earth has a circumference of 40 078 km at the equator.
 (e) The span of the Golden Gate Bridge is 1280 m.
 (f) The Arctic Ocean has an area of 13 986 000 km².

3

- (a) $0.53 = 5.3 \times 10^{-1}$
 (b) $0.0075 = 7.5 \times 10^{-3}$
 (c) $0.000\,004 = 4 \times 10^{-6}$
 (d) $0.000\,020\,01 = 2.001 \times 10^{-5}$

4

- (a) $2.03 \times 10^{-4} = 0.000\,203$
 (b) $1.4 \times 10^{-3} = 0.0014$
 (c) $5.67 \times 10^{-5} = 0.000\,0567$

Section 4.3.2

1

- (a) Estimate: $2521 = 2.521 \times 10^3 \simeq 3 \times 10^3$
 $38 = 3.8 \times 10^1 \simeq 4 \times 10^1$.

$$\begin{aligned}\text{So } 2521 \div 38 &\simeq \frac{3 \times 10^3}{4 \times 10^1} = \frac{3}{4} \times 10^{3-1} \\ &= 0.75 \times 10^2 = 75.\end{aligned}$$

Calculate: $2521 \div 38 = 66.3421$ rounded to six significant figures.

- (b) Estimate: $17.85 = 1.785 \times 10^1 \simeq 2 \times 10^1$
 $286.3 = 2.863 \times 10^2 \simeq 3 \times 10^2$.

$$\begin{aligned}\text{So } 17.85 \times 286.3 &\simeq 2 \times 10^1 \times 3 \times 10^2 \\ &= 2 \times 3 \times 10^1 \times 10^2 = 6 \times 10^3 = 6000.\end{aligned}$$

Calculate: $17.85 \times 286.3 = 5110.46$ rounded to six significant figures.

- (c) Estimate: $1452 = 1.452 \times 10^3 \simeq 1.4 \times 10^3$
 $0.0072 = 7.2 \times 10^{-3} \simeq 7 \times 10^{-3}$.

(Notice that in this case it is easier to round to 1.4 than to 1 or 1.5, anticipating the next step, since it is easier to find $1.4 \div 7$ than $1 \div 7$ or $1.5 \div 7$.)

$$\begin{aligned}\text{So } 1452 \div 0.0072 &\simeq \frac{1.4 \times 10^3}{7 \times 10^{-3}} = \frac{1.4}{7} \times 10^{3-(-3)} \\ &= 0.2 \times 10^{3+3} = 0.2 \times 10^6 = 2 \times 10^{-1} \times 10^6 \\ &= 2 \times 10^{-1+6} = 2 \times 10^5 = 200\,000.\end{aligned}$$

Calculate: $1452 \div 0.0072 = 201\,667$ rounded to six significant figures.

(d)

$$\text{Estimate: } 0.0053 = 5.3 \times 10^{-3} \simeq 5 \times 10^{-3}$$

$$0.0078 = 7.8 \times 10^{-3} \simeq 8 \times 10^{-3}$$

$$0.6821 = 6.821 \times 10^{-1} \simeq 7 \times 10^{-1}.$$

So

$$\begin{aligned}0.0053 \times 0.0078 \div 0.6821 &\simeq \frac{5 \times 10^{-3} \times 8 \times 10^{-3}}{7 \times 10^{-1}} \\ &= \frac{5 \times 8 \times 10^{-3+(-3)}}{7 \times 10^{-1}} \\ &= \frac{40}{7} \times 10^{-6-(-1)} \\ &\simeq 6 \times 10^{-5}.\end{aligned}$$

Calculate: $0.0053 \times 0.0078 \div 0.6821$
 $= 6.06069 \times 10^{-5} = 0.000\,060\,6069$ rounded to six significant figures.

(Notice that the calculator gives the answer in scientific notation.)

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